

Question 1

Consider $\sum_{n=1}^{\infty} \frac{n!}{3^n}$.

- (a) Compute a_{n+1}/a_n .
- (b) Decide whether the series converges.
- (c) Explain what the calculation says about factorial growth compared with 3^n .

Question 1 – Solution**Step 1: Identify the term.**

Let $a_n = n!/3^n$. Its terms are positive, so absolute values do not change the ratio.

Step 2: Simplify before taking the limit.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{3^{n+1}} \frac{3^n}{n!} = \frac{n+1}{3}.$$

Step 3: Interpret the ratio.

Thus $L = \lim_{n \rightarrow \infty} (n+1)/3 = \infty > 1$. The terms eventually increase; indeed, $a_{n+1} > a_n$ for $n > 2$. Therefore a_n cannot approach 0.

Conclusion.

The series **diverges** by the Ratio Test (and by the nth-term test). The unbounded ratio shows that $n!$ eventually grows faster than 3^n .

Question 2

Consider $\sum_{n=0}^{\infty} \frac{3^n}{n!}$.

- (a) Apply the Ratio Test and classify the series.
- (b) Recognize the resulting series and find its exact sum.

Question 2 – Solution

Step 1: Form the ratio.

With $a_n = 3^n/n!$,

$$\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{(n+1)!} \frac{n!}{3^n} = \frac{3}{n+1}.$$

Step 2: Take the limit.

$L = \lim_{n \rightarrow \infty} 3/(n+1) = 0 < 1$. The Ratio Test proves absolute convergence.

Step 3: Identify the sum.

Since $e^x = \sum_{n=0}^{\infty} x^n/n!$, substituting $x = 3$ gives

$$\boxed{\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3}.$$

Question 3

Determine whether $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$ converges or diverges.

- (a) Compute and simplify a_{n+1}/a_n .
- (b) Apply the Ratio Test.
- (c) Explain how the limit compares polynomial and exponential growth.

Question 3 – Solution

Step 1: Compute the exact ratio.

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2 2^n}{2^{n+1} n^2} = \frac{1}{2} \left(1 + \frac{1}{n}\right)^2.$$

Step 2: Evaluate the limit.

$$L = \frac{1}{2}(1)^2 = \frac{1}{2} < 1.$$

Conclusion.

The series **converges absolutely**. Its consecutive terms eventually behave as though multiplied by $1/2$. The polynomial factor changes the ratio by a factor tending to 1, while 2^n supplies the decisive factor $1/2$.

Question 4

Analyze $\sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2 5^n}$. Compute the consecutive-term ratio carefully, apply the Ratio Test, and state the convergence classification.

Question 4 – Solution**Step 1: Name the term.**Let $a_n = (2n)! / [(n!)^2 5^n]$.**Step 2: Expand only the new factorial factors.**

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(2n+2)!}{((n+1)!)^2 5^{n+1}} \frac{(n!)^2 5^n}{(2n)!} \\ &= \frac{(2n+2)(2n+1)}{5(n+1)^2} = \frac{2(2n+1)}{5(n+1)}.\end{aligned}$$

Step 3: Take the limit.

$$L = \lim_{n \rightarrow \infty} \frac{2(2n+1)}{5(n+1)} = \frac{4}{5} < 1.$$

Conclusion.The positive-term series **converges**. The factorial expression is $\binom{2n}{n}$, but no approximation is needed.

Question 5

Use the Ratio Test, without Stirling's formula, to determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges or diverges. Show all algebra used to find the ratio limit.

Question 5 – Solution

Step 1: Form and simplify the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \frac{(n+1)n^n}{(n+1)^{n+1}} \\ &= \left(\frac{n}{n+1}\right)^n = \left(1 + \frac{1}{n}\right)^{-n}.\end{aligned}$$

Step 2: Use the standard exponential limit.

Since $(1 + 1/n)^n \rightarrow e$,

$$L = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

Conclusion.

The series **converges**. This gives the decisive geometric-rate factor $1/e$ directly, without an asymptotic formula for $n!$.

Question 6

For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, where $x \in \mathbb{R}$:

- (a) Apply the Ratio Test to the absolute values.
- (b) Find every real x for which the series converges.
- (c) State the exact sum.

Question 6 – Solution**Step 1: Treat x as fixed.**For $a_n = x^n/n!$,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x|}{n+1}.$$

For $x = 0$ the series plainly converges; the same final limit covers every fixed nonzero x .**Step 2: Take the limit.**

$$L = \lim_{n \rightarrow \infty} \frac{|x|}{n+1} = 0 < 1 \quad \text{for every fixed } x \in \mathbb{R}.$$

Conclusion.

The series converges absolutely for all real x , so its interval of convergence is $(-\infty, \infty)$. By the exponential-series identity,

$$\boxed{\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x}.$$

Question 7

Determine whether $\sum_{n=1}^{\infty} \frac{n!}{n^{10}}$ converges or diverges.

- (a) Simplify a_{n+1}/a_n .
- (b) Use its limiting behavior to justify the conclusion.
- (c) Explain why the nth-term test also applies.

Question 7 – Solution

Step 1: Compute the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{10}} \frac{n^{10}}{n!} = (n+1) \left(\frac{n}{n+1} \right)^{10} \\ &= n \left(1 + \frac{1}{n} \right)^{-9}.\end{aligned}$$

Step 2: Evaluate its behavior.

The factor $(1 + 1/n)^{-9} \rightarrow 1$, while $n \rightarrow \infty$. Hence $L = \infty > 1$.

Conclusion.

The series **diverges**. Its terms eventually grow rather than approach zero, so the nth-term divergence test confirms the result. A factorial eventually dominates every fixed power n^{10} .

Question 8

Analyze $\sum_{n=1}^{\infty} \frac{2^{n^2}}{3^{n^2}}$. Rewrite its term using a single base, compute the consecutive-term ratio, and classify the series.

Question 8 – Solution

Step 1: Combine the powers.

$$a_n = (2/3)^{n^2}.$$

Step 2: Compute the ratio.

$$\frac{a_{n+1}}{a_n} = \left(\frac{2}{3}\right)^{(n+1)^2 - n^2} = \left(\frac{2}{3}\right)^{2n+1}.$$

Because $0 < 2/3 < 1$, this expression tends to 0.

Conclusion.

The ratio limit is $L = 0 < 1$, so the series **converges**. Its quadratic exponent makes the terms decay faster than those of $\sum (2/3)^n$.

Question 9

Determine whether $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$ converges or diverges. Reduce the consecutive-term ratio completely before taking its limit.

Question 9 – Solution

Step 1: Form the ratio.

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1}}{((n+1)!)^2} \frac{(n!)^2}{n^n} = \frac{(n+1)^{n-1}}{n^n} \\ &= \frac{1}{n} \left(1 + \frac{1}{n}\right)^{n-1}.\end{aligned}$$

Step 2: Separate the limiting factors.

We have $(1 + 1/n)^{n-1} \rightarrow e$ and $1/n \rightarrow 0$. Therefore $L = 0$.

Conclusion.

Since $L = 0 < 1$, the positive-term series **converges**. The simplified ratio resolves the competition between n^n and $(n!)^2$ without guesswork.

Question 10

Use the Ratio Test to analyze $\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$. Use logarithms and a standard logarithmic expansion to evaluate the ratio limit.

Question 10 – Solution

Step 1: Describe the term logarithmically.

Let $a_n = (n/(n+1))^{n^2}$. Then

$$\log a_n = -n^2 \log \left(1 + \frac{1}{n} \right).$$

Using $\log(1+t) = t - t^2/2 + O(t^3)$,

$$\log a_n = -n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + O(n^{-3}) \right) = -n + \frac{1}{2} + O(n^{-1}).$$

Step 2: Find the ratio limit.

$$\log \left(\frac{a_{n+1}}{a_n} \right) = \log a_{n+1} - \log a_n = -1 + O(n^{-1}) \rightarrow -1.$$

Exponentiating gives $L = e^{-1} < 1$.

Conclusion.

The series **converges**. More specifically, $a_n \sim e^{1/2}e^{-n}$, so its terms have geometric-scale decay.