

Question 1

Let $a_n = \frac{3n-1}{n+2}$ for $n \geq 1$.

- (a) Find $L = \lim_{n \rightarrow \infty} a_n$.
- (b) Solve $|a_n - L| < 0.01$ exactly and determine the smallest integer N such that the inequality holds for every $n \geq N$.

Question 1 – Solution**Step 1: Find the candidate limit.**

The numerator and denominator have the same degree. Divide both by n :

$$a_n = \frac{3 - 1/n}{1 + 2/n} \rightarrow \frac{3}{1} = 3.$$

Since $1/n \rightarrow 0$ and $2/n \rightarrow 0$, the quotient law gives $L = 3$.

Step 2: Express the error exactly.

Rather than approximate numerically, subtract the limit and simplify:

$$|a_n - 3| = \left| \frac{3n - 1}{n + 2} - 3 \right| = \frac{7}{n + 2}.$$

Step 3: Solve the strict accuracy inequality.

Therefore

$$\frac{7}{n + 2} < 0.01 = \frac{1}{100} \iff n > 698.$$

Thus every integer $n \geq 699$ satisfies the requirement.

Step 4: Verify minimality.

The smallest possible integer is $N = 699$. At the preceding index, $n = 698$, the error is $7/700 = 0.01$, which equals the tolerance and is not strictly less than it. Hence no smaller N works.

Question 2

Let $a_n = (-1)^n \left(1 + \frac{1}{n}\right)$ for $n \geq 1$.

- (a) Show that (a_n) is bounded.
- (b) Find the limits of the even and odd subsequences (a_{2k}) and (a_{2k-1}) .
- (c) Use those limits to decide whether (a_n) converges and to evaluate the claim, “Every bounded sequence converges.”

Question 2 – Solution**Step 1: Prove boundedness.**

For $n \geq 1$, $1 < 1 + 1/n \leq 2$. Since $|(-1)^n| = 1$,

$$|a_n| = 1 + \frac{1}{n} \leq 2.$$

Thus $-2 \leq a_n \leq 2$, and the sequence is bounded.

Step 2: Analyze the even subsequence.

For even indices,

$$a_{2k} = 1 + \frac{1}{2k} \longrightarrow 1.$$

Step 3: Analyze the odd subsequence.

For odd indices,

$$a_{2k-1} = -1 - \frac{1}{2k-1} \longrightarrow -1.$$

Step 4: Apply the subsequence criterion.

If (a_n) converged to L , every subsequence would also converge to L . The even and odd subsequences instead approach 1 and -1 , respectively. Since these limits differ, (a_n) diverges.

Step 5: Evaluate the student's claim.

This bounded divergent sequence is a counterexample to the claim that every bounded sequence converges. Boundedness together with monotonicity, however, would guarantee convergence by the Monotone Convergence Theorem.

Question 3

Let $a_n = \frac{n}{n+1}$ for $n \geq 1$.

- (a) Prove algebraically (without derivatives) that (a_n) is strictly increasing.
- (b) Find an upper bound and use the Monotone Convergence Theorem to explain why a limit exists.
- (c) Find the limit and express the error $|a_n - L|$ exactly.

Question 3 – Solution**Step 1: Prove monotonicity algebraically.**

Compare consecutive terms by subtraction:

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{1}{(n+1)(n+2)} > 0.$$

The denominator is positive for $n \geq 1$, so $a_{n+1} - a_n > 0$ and (a_n) is strictly increasing.**Step 2: Find an upper bound.**

Rewrite the term as

$$a_n = 1 - \frac{1}{n+1} < 1,$$

Thus $a_n < 1$ for every n , so the sequence is bounded above by 1.**Step 3: Establish convergence and calculate the limit.**

An increasing sequence bounded above converges by the Monotone Convergence Theorem. Using the rewritten formula,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1.$$

Step 4: Express the error exactly.Since $a_n < 1$, the exact error is $|a_n - 1| = 1 - a_n = 1/(n+1)$, which tends to zero.

Question 4

Let $a_n = \left(1 + \frac{2}{n}\right)^n$.

- (a) Interpret a_n as the balance produced by compounding a principal of 1 at a total rate of 2 over one time unit, using n periods.
- (b) Evaluate $\lim_{n \rightarrow \infty} a_n$ using the standard exponential limit.
- (c) Give a decimal approximation of the limiting balance.

Question 4 – Solution**Step 1: Interpret the compounding model.**

The principal is $P = 1$. Dividing the total rate 2 into n equal periods gives rate $2/n$ per period; applying that growth factor n times produces $1(1 + 2/n)^n$.

Step 2: Rewrite in the standard exponential form.

Put $m = n/2$. Then $m \rightarrow \infty$ and

$$a_n = \left[\left(1 + \frac{1}{m} \right)^m \right]^2.$$

Step 3: Apply the standard limit.

Using $\lim_{m \rightarrow \infty} (1 + 1/m)^m = e$ and continuity of the squaring function gives

$$\lim_{n \rightarrow \infty} a_n = e^2 \approx 7.38906.$$

Step 4: Interpret and verify.

This agrees with the continuous-compounding formula $Pe^{rt} = e^2$ for $P = 1$, $r = 2$, and $t = 1$. The standard exponential limit supplies the proof.

Question 5

Let $a_n = \sin(n\pi/2)$ for $n \geq 1$.

- (a) Make a table giving a_n for each residue class of n modulo 4, and list the first eight terms.
- (b) Determine whether the sequence converges. Justify your answer using subsequences.
- (c) Explain whether changing or rearranging finitely many initial terms can change convergence.

Question 5 – Solution**Step 1: Use periodicity.**

Since adding 2π does not change sine and increasing n by 4 adds 2π to $n\pi/2$, the values repeat every four indices:

$n \pmod{4}$	1	2	3	0
a_n	1	0	-1	0

Thus the first eight terms are $1, 0, -1, 0, 1, 0, -1, 0$.

Step 2: Select decisive subsequences.

In particular,

$$a_{4k+1} = 1 \quad \text{and} \quad a_{4k+3} = -1$$

for all applicable k . The first subsequence converges to 1, while the second converges to -1 .

Step 3: Conclude divergence.

If (a_n) converged, all its subsequences would share the same limit. These constant subsequences have different limits, so (a_n) diverges.

Step 4: Analyze finite modifications.

Changing, deleting, or rearranging only finitely many initial terms cannot change convergence because the definition concerns all terms beyond some index. No finite modification removes the infinitely recurring values 1 and -1 . Even an arbitrary permutation of all these terms still contains infinitely many occurrences of both values, so it cannot converge.

Question 6

Let $a_n = n^{1/n}$ for $n \geq 1$.

- (a) Use logarithms to find $\lim_{n \rightarrow \infty} a_n$.
- (b) For which integers n is $a_n > 1$?
- (c) Use the continuous extension $f(x) = x^{1/x}$ to explain why the terms first rise and then decrease, even though their limit is 1.

Question 6 – Solution**Step 1: Transform the variable exponent.**

Because $a_n = n^{1/n} > 0$, logarithms are valid. Let $b_n = \ln a_n$. Then

$$b_n = \ln(n^{1/n}) = \frac{\ln n}{n}.$$

Step 2: Evaluate the logarithmic limit.

Extend the quotient to $g(x) = \ln x/x$. It has the indeterminate form ∞/∞ , and L'Hopital's Rule gives

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x} = \lim_{x \rightarrow \infty} \frac{1/x}{1} = 0.$$

Restricting this continuous limit to integer values gives $b_n \rightarrow 0$. By continuity of the exponential function, $a_n = e^{b_n} \rightarrow e^0 = 1$.

Step 3: Determine when the terms exceed the limit.

For every integer $n > 1$, the base $n > 1$ and exponent $1/n > 0$, so $a_n > 1$; also $a_1 = 1$. A sequence may approach a limit entirely from one side, so convergence to 1 does not require later terms to equal or cross 1.

Step 4: Explain the rise and fall.

For the continuous extension $f(x) = x^{1/x}$, logarithmic differentiation gives

$$\frac{f'(x)}{f(x)} = \frac{1 - \ln x}{x^2}.$$

Because $f(x) > 0$, the sign of $f'(x)$ is the sign of $1 - \ln x$. Thus f increases for $x < e$ and decreases for $x > e$. Among integer indices, the maximum occurs at $n = 3$, after which the sequence decreases toward 1 while remaining above it.

Question 7

Let $a_n = \frac{\ln n}{\sqrt{n}}$ for $n \geq 1$.

- (a) Use a continuous extension and L'Hopital's Rule to find the limit.
- (b) Explain why taking logarithms of a_n is not the most direct method.
- (c) Use the derivative of $f(x) = \ln x / \sqrt{x}$ to explain the initial increase of the sequence and identify where the eventual decrease begins.

Question 7 – Solution

Step 1: Choose a continuous extension.

Let $f(x) = \ln x / \sqrt{x}$ for $x \geq 1$. Both numerator and denominator tend to infinity, so the quotient has the indeterminate form ∞/∞ and L'Hopital's Rule applies:

$$\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} = \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0.$$

Restricting the continuous limit to integer inputs gives $a_n \rightarrow 0$.

Step 2: Compare possible methods.

Taking $\ln(a_n) = \ln(\ln n) - \frac{1}{2} \ln n$ would introduce an $\infty - \infty$ expression and still require comparing logarithmic growth. Applying L'Hopital's Rule directly to the original quotient is cleaner.

Step 3: Determine where the sequence increases and decreases.

Differentiate the extension:

$$f'(x) = x^{-3/2} \left(1 - \frac{1}{2} \ln x \right).$$

Since $x^{-3/2} > 0$, $f'(x) > 0$ for $x < e^2$, equals 0 at $e^2 \approx 7.39$, and is negative afterward. The integer terms rise through $n = 7$ and then decrease very slowly. This initial increase is consistent with the eventual limit of zero.

Question 8

Define $a_1 = 0$ and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$.

- (a) Prove by induction that $0 \leq a_n < 2$ for every n .
- (b) Prove that (a_n) is increasing, and conclude that it converges.
- (c) Find its limit by solving the appropriate fixed-point equation, and explain why the other algebraic root is inadmissible.

Question 8 – Solution**Step 1: Prove an invariant bound.**

We prove $0 \leq a_n < 2$ by induction. The base case holds because $a_1 = 0$. Assume $0 \leq a_n < 2$. Then

$$0 < 2 + a_n < 4 \implies 0 < a_{n+1} = \sqrt{2 + a_n} < 2.$$

Thus the bound holds for a_{n+1} , completing the induction.

Step 2: Prove monotonicity.

Use induction again. Since $a_2 = \sqrt{2} > a_1$, the base case holds. If $a_n > a_{n-1}$, the square-root function is strictly increasing, so

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{2 + a_{n-1}} = a_n.$$

Thus (a_n) is increasing. Together with the upper bound 2, the Monotone Convergence Theorem guarantees $a_n \rightarrow L$ for some $0 \leq L \leq 2$.

Step 3: Solve the fixed-point equation.

Only after proving convergence may we pass to the limit in the recurrence. Continuity of the square root gives

$$L = \sqrt{2 + L} \implies L^2 - L - 2 = 0 \implies (L - 2)(L + 1) = 0.$$

The algebraic candidates are $L = 2$ and $L = -1$. Because every term and hence its limit are non-negative, $L = -1$ is inadmissible. Therefore $L = 2$. Solving this equation alone would identify only possible limits; boundedness and monotonicity are what prove a limit actually exists.

Question 9

Let $a_n = \frac{(n!)^{1/n}}{n}$ for $n \geq 1$.

- (a) Apply Stirling's formula $n! \sim \sqrt{2\pi n}(n/e)^n$ to rewrite a_n as a dominant constant times a factor tending to 1.
- (b) Find $\lim_{n \rightarrow \infty} a_n$ and give its decimal value.
- (c) Explain why the first dozen terms alone do not reveal the limit accurately.

Question 9 – Solution**Step 1: Insert Stirling's approximation.**

Stirling's formula states

$$n! = \sqrt{2\pi n} \left(\frac{n}{e}\right)^n (1 + o(1)).$$

Step 2: Apply the n th root and normalize.All quantities are positive, so taking n th roots and dividing by n gives

$$a_n = \frac{1}{n} (2\pi n)^{1/(2n)} \frac{n}{e} (1 + o(1))^{1/n} = \frac{1}{e} (2\pi n)^{1/(2n)} (1 + o(1))^{1/n}.$$

Step 3: Evaluate the correction factors.

The extra factors tend to 1. For the first factor, take logarithms:

$$\ln \left((2\pi n)^{1/(2n)} \right) = \frac{\ln(2\pi n)}{2n} \rightarrow 0,$$

so exponentiating gives $(2\pi n)^{1/(2n)} \rightarrow 1$. Also, if $u_n = o(1)$, then $(1 + u_n)^{1/n} \rightarrow 1$ because its logarithm is $\ln(1 + u_n)/n \rightarrow 0$. Consequently,

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{e} \approx 0.367879.$$

Step 4: Interpret the numerical evidence.

At $n = 12$, $a_{12} \approx 0.4407$, still about 0.0729 above $1/e$. The slowly decaying correction factor $(2\pi n)^{1/(2n)}$ explains why a short numerical table is suggestive but cannot identify the hidden constant reliably.

Question 10

Let $a_n = \cos(1/n)$, where angles are measured in radians.

- (a) Find $\lim_{n \rightarrow \infty} a_n$ using continuity.
- (b) Use $0 \leq 1 - \cos x \leq x^2/2$ to obtain an explicit upper bound for the error $|a_n - L|$.
- (c) Find a simple integer N that guarantees $|a_n - L| < 10^{-4}$ for every $n \geq N$.

Question 10 – Solution

Step 1: Find the limit by continuity.

Since $1/n \rightarrow 0$ and cosine is continuous at 0,

$$L = \lim_{n \rightarrow \infty} \cos(1/n) = \cos 0 = 1.$$

Step 2: Express and bound the error.

Because $\cos(1/n) \leq 1$, the absolute error simplifies to

$$|a_n - L| = 1 - \cos(1/n).$$

Apply $0 \leq 1 - \cos x \leq x^2/2$ with $x = 1/n$:

$$0 \leq |a_n - 1| \leq \frac{1}{2n^2}.$$

Step 3: Solve for a sufficient index.

It is enough to make the upper bound smaller than the desired tolerance:

$$\frac{1}{2n^2} < 10^{-4} \iff n^2 > 5000 \iff n > \sqrt{5000} \approx 70.71.$$

Step 4: State the guarantee precisely.

Thus $N = 71$ guarantees $|a_n - 1| < 10^{-4}$ for every $n \geq 71$, because $1/(2n^2)$ decreases with n . This is a sufficient value obtained from an upper bound; it need not be the smallest possible N for the exact cosine error.