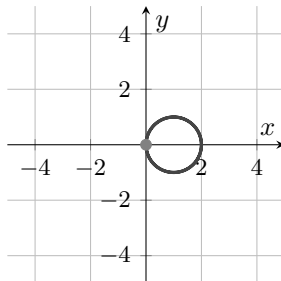
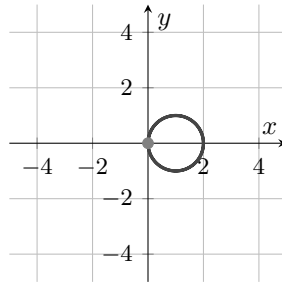


Question 1**Problem**

Find the tangent line to $r = 2 \cos \theta$ at $\theta = \pi/4$.



Question 1 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

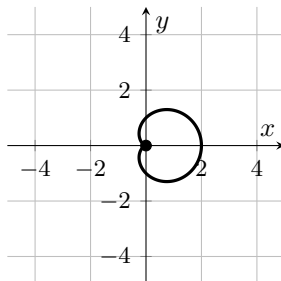
3. The point is $(1, 1)$.

4. With $r' = -2 \sin \theta$, the polar slope formula gives 0, so the tangent is $y = 1$.

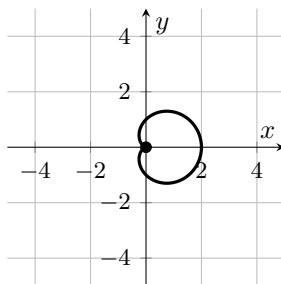
Question 2

Problem

Find all horizontal tangents of the cardioid $r = 1 + \cos \theta$ away from the pole.



Question 2 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

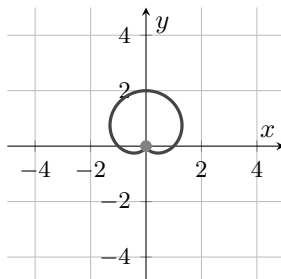
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

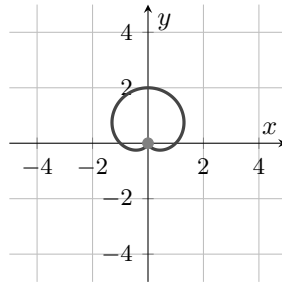
3. The numerator $r' \sin \theta + r \cos \theta = \cos \theta + \cos 2\theta$ vanishes.
4. This gives $\cos \theta = 1/2$ or -1 , and excluding the pole leaves $\theta = \pi/3, 5\pi/3$. At these angles $dx/d\theta \neq 0$, and the points are $(3/4, \pm 3\sqrt{3}/4)$. The tangent lines are $y = \pm 3\sqrt{3}/4$.

Question 3**Problem**

A student claims that $\theta = 0$ gives the rightmost point and a vertical tangent of $r = 1 + \sin \theta$. Test both parts of the claim.



Question 3 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

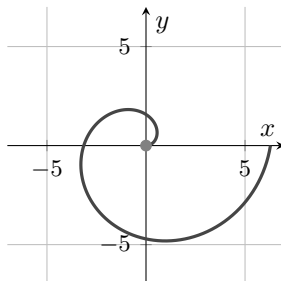
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

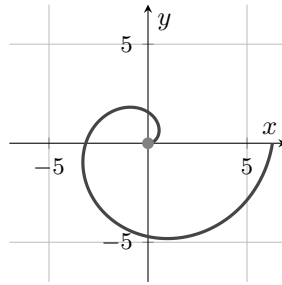
3. At $\theta = 0$, the point is $(1, 0)$ and the slope is 1, so the tangent is not vertical.
4. Also $x'(0) = 1 > 0$, so x is still increasing; the point is not rightmost.
5. Both parts are false.

Question 4**Problem**

The spiral $r = \theta$ reaches $\theta = \pi$. Find its tangent slope there.



Question 4 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

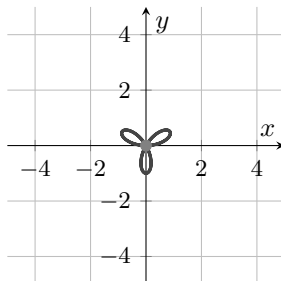
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

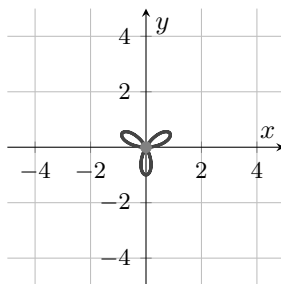
3. With $r' = 1$, the slope is $(\sin \pi + \pi \cos \pi)/(\cos \pi - \pi \sin \pi) = \boxed{\pi}$.

Question 5**Problem**

Find the tangent directions at the pole for the rose $r = \sin 3\theta$.



Question 5 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

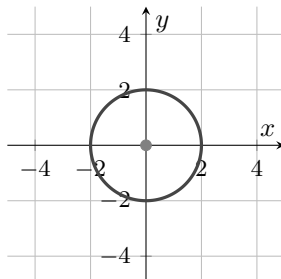
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The pole occurs when $3\theta = k\pi$, so the tangent rays satisfy $\theta = k\pi/3$.
4. The distinct unoriented lines are

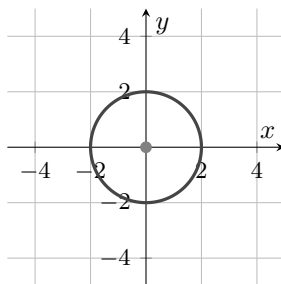
$$\theta = 0, \quad \theta = \pi/3, \quad \theta = 2\pi/3.$$

Question 6**Problem**

At an intersection of $r = 2$ and $r = 4 \cos \theta$ in the upper half-plane, find the acute angle between the two curves.



Question 6 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

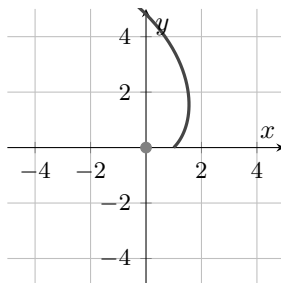
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $\theta = \pi/3$, the circle $r = 2$ has tangent direction $5\pi/6$.
4. The second circle has slope $1/\sqrt{3}$, hence tangent direction $\pi/6$.
5. The smaller angle between the tangent lines is $\boxed{\pi/3}$.

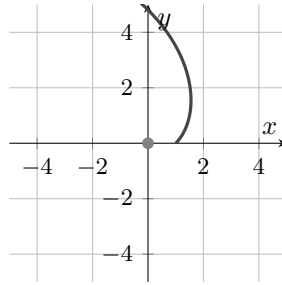
Question 7

Problem

For $r = e^\theta$, show that the angle between the radius vector and tangent is constant.



Question 7 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

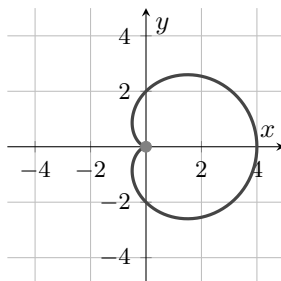
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. In polar components, tangent has radial component $r' = r$ and transverse component r .
4. Thus it makes angle $\arctan(r/r) = \boxed{\pi/4}$ with the radius vector.

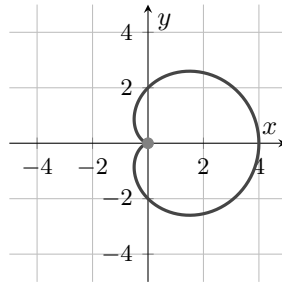
Question 8

Problem

Find a value of a so that $r = a + 2 \cos \theta$ has tangent slope 1 at $\theta = \pi/2$.



Question 8 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

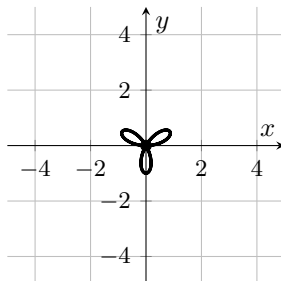
Test numerator and denominator separately when locating horizontal or vertical tangents.

3. At $\pi/2$, the slope is $2/a$.

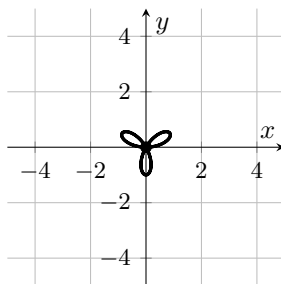
4. Setting $2/a = 1$ gives $\boxed{a = 2}$.

Question 9**Problem**

A student cancels r before using the polar slope formula at a pole. Explain the danger.



Question 9 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

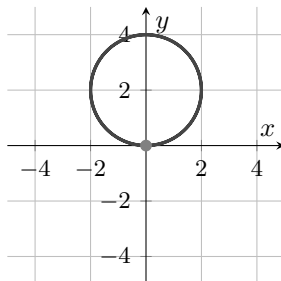
$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

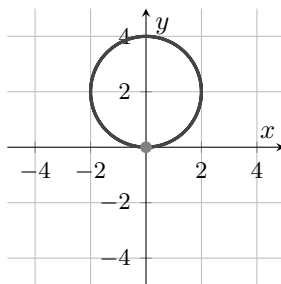
3. At $r = 0$, direct division by r is undefined. Valid cancellation where $r \neq 0$ can help compute a limit, but does not by itself justify substitution at the pole.
4. One must use a limit or inspect the parameter values producing the pole.

Question 10**Problem**

For the circle $r = 4 \sin \theta$, find the tangent at its highest point using polar calculus.



Question 10 – Solution



Solution

1. Write the polar curve parametrically as

$$x(\theta) = r(\theta) \cos \theta, \quad y(\theta) = r(\theta) \sin \theta.$$

Differentiation gives

$$\frac{dx}{d\theta} = r' \cos \theta - r \sin \theta, \quad \frac{dy}{d\theta} = r' \sin \theta + r \cos \theta.$$

2. Wherever $dx/d\theta \neq 0$, compute

$$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}.$$

Test numerator and denominator separately when locating horizontal or vertical tangents.

3. The highest point occurs at $\theta = \pi/2$, giving $(0, 4)$.
4. The slope numerator is 0 and denominator -4 , so the tangent is horizontal: $y = 4$.