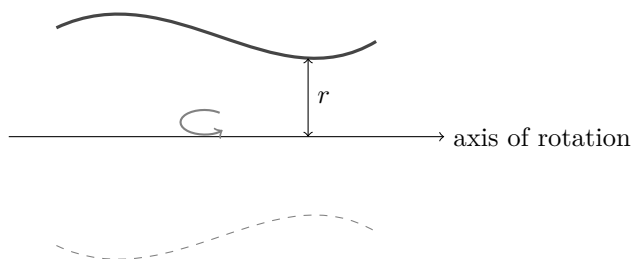


Calculus II Section 3-5: Surface Area with Parametric Equations

Question 1

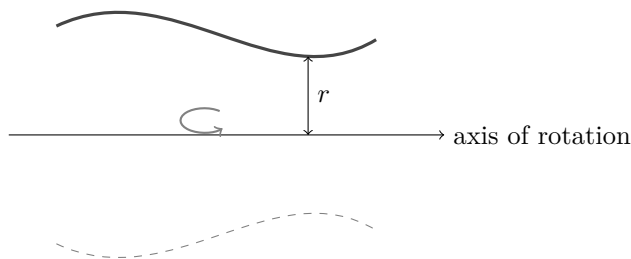
Problem

Rotate the line segment $x = t$, $y = 1 - t$, $0 \leq t \leq 1$, about the x -axis. Find the surface area and identify the surface.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 1 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 1, \quad y'(t) = -1.$$

Therefore,

$$ds = \sqrt{1^2 + (-1)^2} dt = \sqrt{2} dt.$$

2. The distance from (x, y) to the x -axis is $y = 1 - t$, which is nonnegative on $0 \leq t \leq 1$.
3. Apply the surface-area formula:

$$\begin{aligned} S &= 2\pi \int_0^1 y \, ds \\ &= 2\pi\sqrt{2} \int_0^1 (1-t) \, dt \\ &= 2\pi\sqrt{2} \left[t - \frac{t^2}{2} \right]_0^1 = \boxed{\pi\sqrt{2}}. \end{aligned}$$

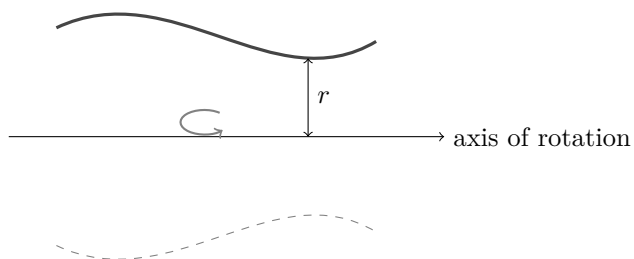
4. The segment runs from $(0, 1)$ to $(1, 0)$. Rotating it about the x -axis produces a right circular cone with radius 1, height 1, and slant height $\sqrt{2}$. The result agrees with $\pi r \ell = \pi\sqrt{2}$.

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 2

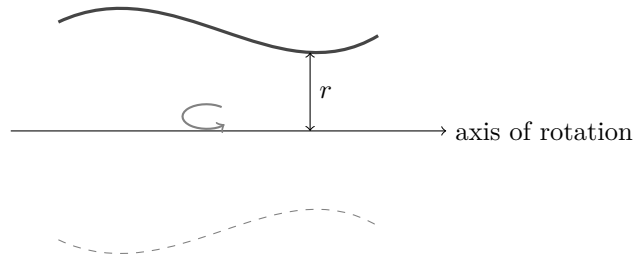
Problem

The upper semicircle $x = R \cos t$, $y = R \sin t$, $0 \leq t \leq \pi$, rotates about the x -axis. Recover the sphere's area.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 2 – Solution



Solution

1. Differentiate:

$$x'(t) = -R \sin t, \quad y'(t) = R \cos t.$$

Thus

$$ds = \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt = R dt.$$

2. On $0 \leq t \leq \pi$, the generating semicircle lies above the x -axis, so its radius of rotation is

$$y = R \sin t \geq 0.$$

This semicircle generates the sphere exactly once.

3. Substitute into $S = 2\pi \int y ds$:

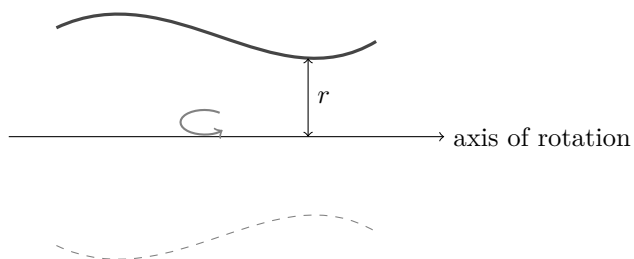
$$\begin{aligned} S &= 2\pi \int_0^\pi (R \sin t)(R dt) \\ &= 2\pi R^2 [-\cos t]_0^\pi \\ &= 2\pi R^2 (2) = \boxed{4\pi R^2}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 3

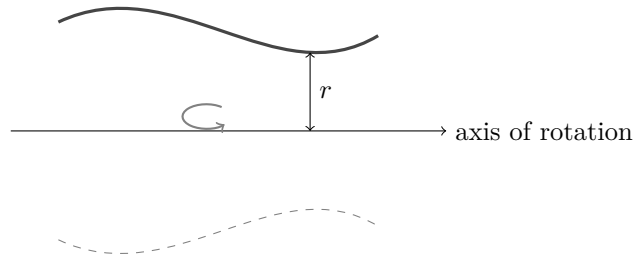
Problem

A circle $x = 3 + \cos t$, $y = \sin t$ rotates about the y -axis. Find the torus area without citing a memorized formula.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 3 – Solution



Solution

1. The interval $0 \leq t \leq 2\pi$ traces the generating circle once. Differentiate:

$$x'(t) = -\sin t, \quad y'(t) = \cos t.$$

Hence

$$ds = \sqrt{\sin^2 t + \cos^2 t} dt = dt.$$

2. The circle lies between $x = 2$ and $x = 4$, entirely to the right of the y -axis. Therefore, the radius of rotation is the nonnegative distance

$$x = 3 + \cos t.$$

3. Apply $S = 2\pi \int x ds$:

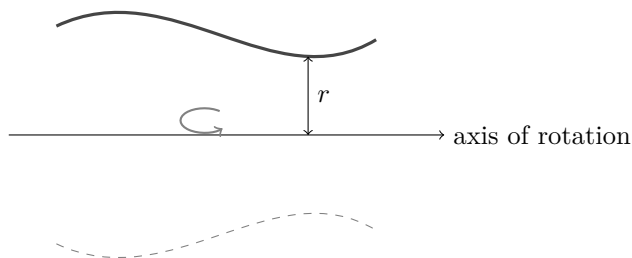
$$\begin{aligned} S &= 2\pi \int_0^{2\pi} (3 + \cos t) dt \\ &= 2\pi [3t + \sin t]_0^{2\pi} \\ &= 2\pi(6\pi) = \boxed{12\pi^2}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

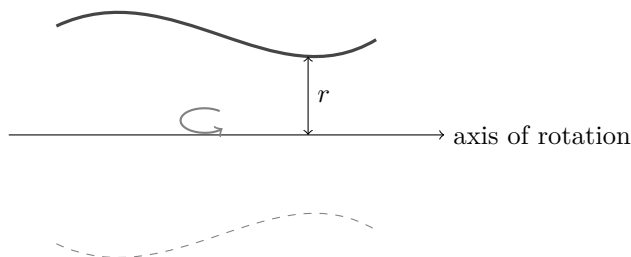
Question 4

Problem

Set up, but do not evaluate, the area formed by rotating $x = t^2$, $y = t^3$, $0 \leq t \leq 1$, about the y -axis.



Question 4 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = 2t, \quad y'(t) = 3t^2.$$

2. Compute the arc-length element:

$$\begin{aligned} ds &= \sqrt{(2t)^2 + (3t^2)^2} dt \\ &= \sqrt{4t^2 + 9t^4} dt = |t|\sqrt{4 + 9t^2} dt. \end{aligned}$$

Since $0 \leq t \leq 1$, this becomes $ds = t\sqrt{4 + 9t^2} dt$.

3. Rotation is about the y -axis, so the radius is the distance $x = t^2$.
4. Therefore, without evaluating,

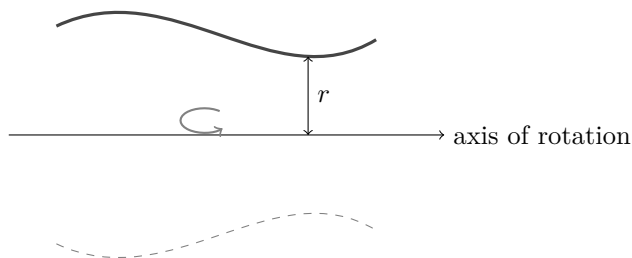
$$\begin{aligned} S &= 2\pi \int_0^1 x(t) ds \\ &= \boxed{2\pi \int_0^1 t^3 \sqrt{4 + 9t^2} dt}. \end{aligned}$$

Calculus II Section 3-5: Surface Area with Parametric Equations

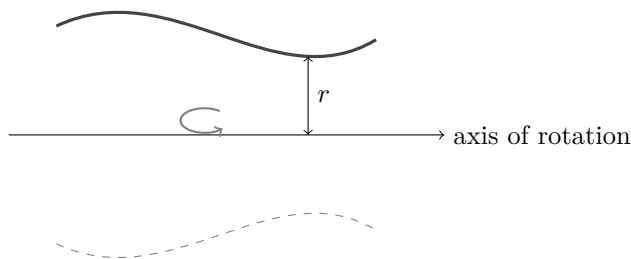
Question 5

Problem

Why is rotating $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$, about the x -axis a bad direct use of the surface formula?



Question 5 – Solution



Solution

1. The interval $0 \leq t \leq \pi$ traces the upper semicircle, while $\pi \leq t \leq 2\pi$ traces the lower semicircle.
2. When either semicircle is rotated about the x -axis, it generates the entire unit sphere. A point (x, y) and its reflection $(x, -y)$ produce the same circle of revolution because both are the same distance $|y|$ from the axis.
3. If the full interval is used with the correct radius $|y|$, then $ds = dt$ and

$$2\pi \int_0^{2\pi} |\sin t| dt = 8\pi,$$

which is twice the unit sphere's area. The formula counts the repeated sweep.

4. If one incorrectly uses signed $y = \sin t$ instead, the upper and lower contributions cancel and give zero, which is also not physical surface area.
5. Use one generating semicircle only, for example

$$S = 2\pi \int_0^\pi \sin t dt = \boxed{4\pi}.$$

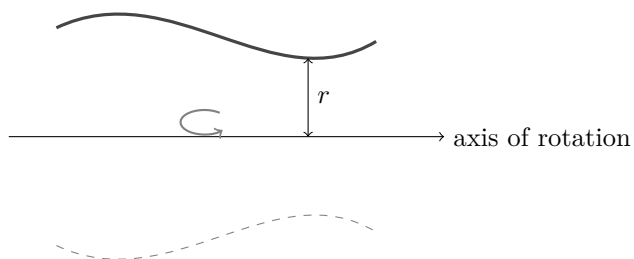
Thus the full-circle interval is unsuitable because it double-covers the surface.

Calculus II Section 3-5: Surface Area with Parametric Equations

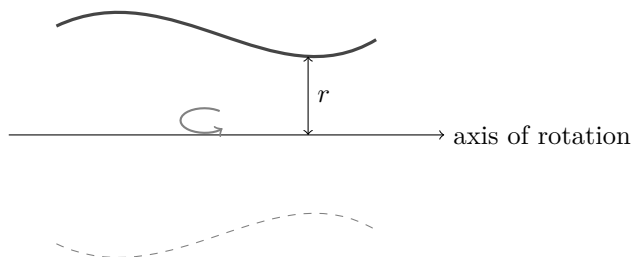
Question 6

Problem

Rotate $x = e^t$, $y = e^{-t}$, $0 \leq t \leq \ln 2$, about the x -axis. Write the exact surface-area integral in simplest form.



Question 6 – Solution



Solution

1. Differentiate:

$$x'(t) = e^t, \quad y'(t) = -e^{-t}.$$

Therefore,

$$ds = \sqrt{e^{2t} + e^{-2t}} dt.$$

2. The curve lies above the x -axis because $y = e^{-t} > 0$, so the radius of rotation is $y = e^{-t}$.
3. Multiply the radius by ds and simplify:

$$\begin{aligned} y ds &= e^{-t} \sqrt{e^{2t} + e^{-2t}} dt \\ &= \sqrt{e^{-2t}(e^{2t} + e^{-2t})} dt \\ &= \sqrt{1 + e^{-4t}} dt. \end{aligned}$$

4. Hence the requested exact integral is

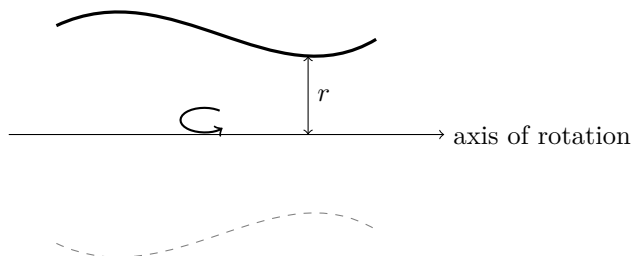
$$S = 2\pi \int_0^{\ln 2} \sqrt{1 + e^{-4t}} dt.$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 7

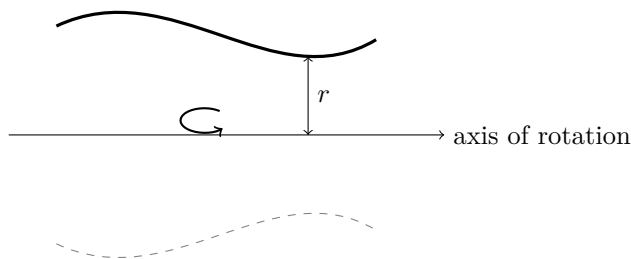
Problem

A cylinder is generated parametrically by rotating the horizontal segment $x = t, y = h, 0 \leq t \leq L$, about the x -axis, where $h, L > 0$. Derive its lateral area.



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 7 – Solution



Solution

1. Differentiate the horizontal segment:

$$x'(t) = 1, \quad y'(t) = 0.$$

Hence

$$ds = \sqrt{1^2 + 0^2} dt = dt.$$

2. Every point of the segment is the constant distance h from the x -axis, assuming $h \geq 0$. Thus the radius of rotation is h .
3. Apply the surface-area formula:

$$\begin{aligned} S &= 2\pi \int_0^L h dt \\ &= 2\pi h[t]_0^L = \boxed{2\pi hL}. \end{aligned}$$

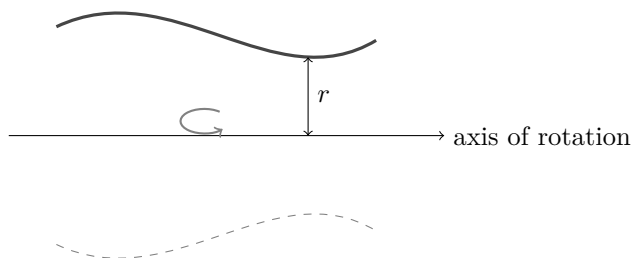
4. This is circumference times length: $(2\pi h)L$, which is precisely the lateral area of a cylinder of radius h and length L .

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 8

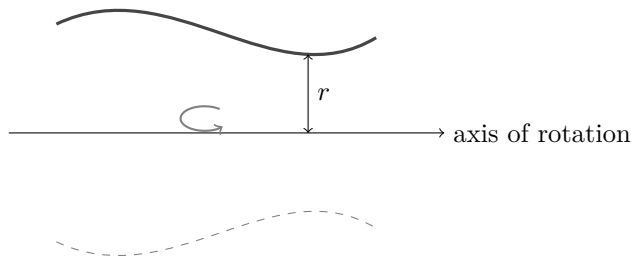
Problem

A curve crosses the axis of rotation. What precaution is needed in $S = 2\pi \int y \, ds$?



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 8 – Solution



Solution

1. In $S = 2\pi \int y \, ds$, the factor multiplying 2π must be the radius of rotation. A radius is a distance and therefore cannot be negative.
2. When the axis is the x -axis, the distance from $(x(t), y(t))$ to that axis is

$$r(t) = |y(t)|,$$

not signed $y(t)$.

3. The correct general formula is therefore

$$S = 2\pi \int_a^b |y(t)| \sqrt{x'(t)^2 + y'(t)^2} \, dt.$$

Equivalently, split the integral at every parameter value where $y(t) = 0$ and use the appropriate sign on each subinterval.

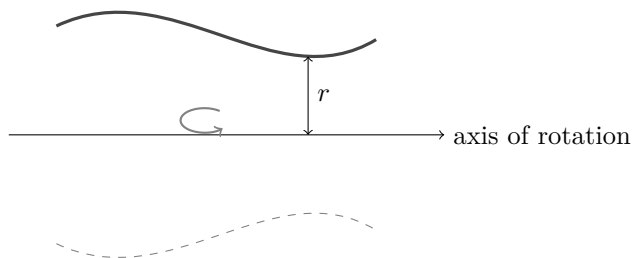
4. Without the absolute value, contributions below the axis would be negative and could cancel contributions above the axis. Also check whether portions on opposite sides of the axis generate the same physical surface; if they do, restrict the interval to prevent double-counting.

Calculus II Section 3-5: Surface Area with Parametric Equations

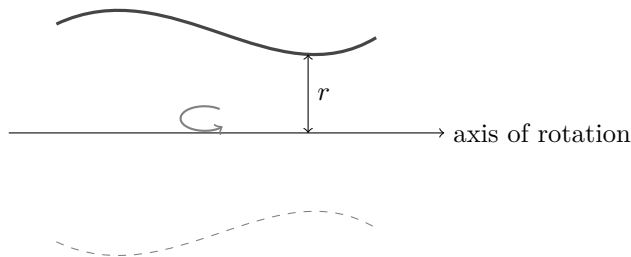
Question 9

Problem

Rotate the astroid quadrant $x = \cos^3 t, y = \sin^3 t, 0 \leq t \leq \pi/2$, about the x -axis. Set up a simplified integral.



Question 9 – Solution



Solution

1. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

2. As in the astroid arc-length calculation,

$$ds = 3 |\sin t \cos t| dt.$$

Both factors are nonnegative on $0 \leq t \leq \pi/2$, so

$$ds = 3 \sin t \cos t dt.$$

3. The radius about the x -axis is $y = \sin^3 t$. Therefore,

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} y ds \\ &= 6\pi \int_0^{\pi/2} \sin^4 t \cos t dt. \end{aligned}$$

4. Let $u = \sin t$, so $du = \cos t dt$ and $0 \leq u \leq 1$:

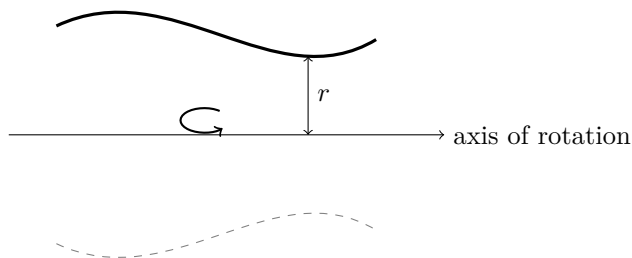
$$S = 6\pi \int_0^1 u^4 du = 6\pi \left[\frac{u^5}{5} \right]_0^1 = \boxed{\frac{6\pi}{5}}.$$

Calculus II Section 3-5: Surface Area with Parametric Equations

Question 10

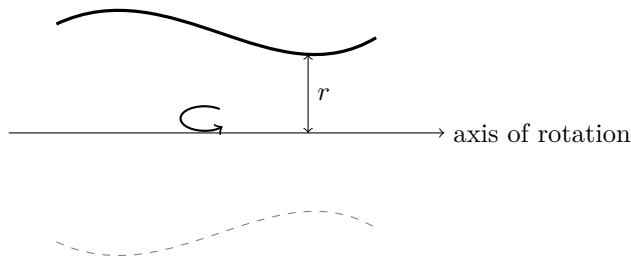
Problem

Two curves have the same length and the same arc-length-weighted average distance from an axis, and each sweeps its surface exactly once. Must their surfaces of revolution have equal area?



Calculus II Section 3-5: Surface Area with Parametric Equations

Question 10 – Solution



Solution

1. Let $r(s)$ denote the nonnegative distance from a point on a generating curve to the axis, where s is arc length. Its arc-length-weighted average distance is

$$r_{\text{avg}} = \frac{1}{L} \int_C r \, ds.$$

2. Multiply by L :

$$\int_C r \, ds = L r_{\text{avg}}.$$

The surface area of revolution is therefore

$$S = 2\pi \int_C r \, ds = 2\pi L r_{\text{avg}}.$$

3. If two curves have the same length L and the same arc-length-weighted average radius, this formula gives the same value of S for both.
4. Thus the answer is , provided each generating curve sweeps its surface exactly once. If a parametrization retraces the curve or distinct pieces generate the same surface, the integral counts multiplicity and the comparison requires removing that double-counting first.