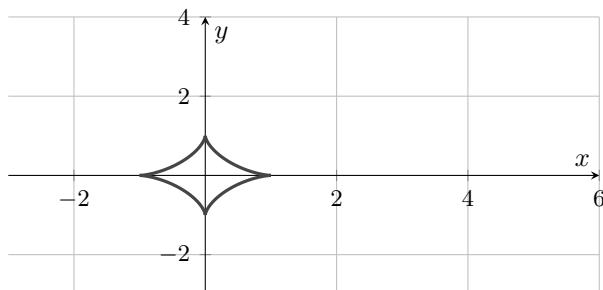


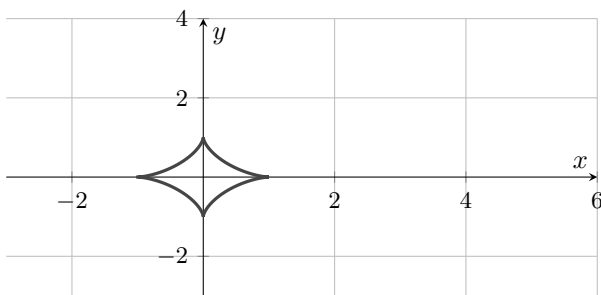
Question 1

Problem

Find the length of the astroid $x = \cos^3 t$, $y = \sin^3 t$.



Question 1 – Solution



Solution

1. The astroid is symmetric about both coordinate axes, so compute the length in the first quadrant, $0 \leq t \leq \pi/2$, and multiply by four.

2. Differentiate:

$$x'(t) = -3 \cos^2 t \sin t, \quad y'(t) = 3 \sin^2 t \cos t.$$

3. Simplify the speed:

$$\begin{aligned} \sqrt{x'^2 + y'^2} &= 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} \\ &= 3\sqrt{\sin^2 t \cos^2 t (\cos^2 t + \sin^2 t)} \\ &= 3|\sin t \cos t|. \end{aligned}$$

On $0 \leq t \leq \pi/2$, both factors are nonnegative, so the speed is $3 \sin t \cos t$.

4. The first-quadrant length is

$$L_1 = 3 \int_0^{\pi/2} \sin t \cos t \, dt = \frac{3}{2} [\sin^2 t]_0^{\pi/2} = \frac{3}{2}.$$

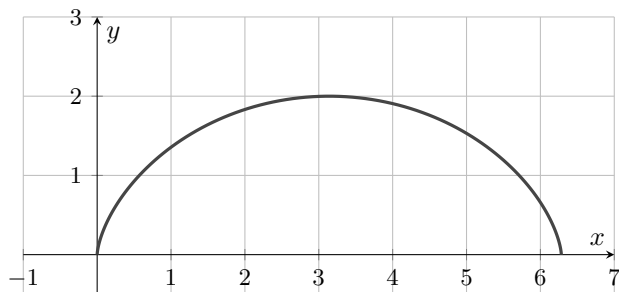
5. Therefore, the total length is

$$L = 4L_1 = 4 \left(\frac{3}{2} \right) = 6.$$

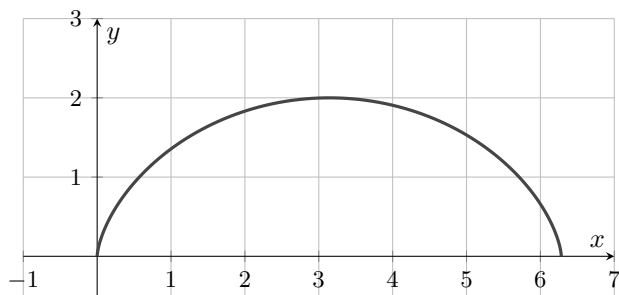
Question 2

Problem

Find the length of one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Question 2 – Solution



Solution

1. Differentiate the parametric equations:

$$x'(t) = 1 - \cos t, \quad y'(t) = \sin t.$$

2. Compute and simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= (1 - \cos t)^2 + \sin^2 t \\ &= 1 - 2 \cos t + \cos^2 t + \sin^2 t \\ &= 2 - 2 \cos t = 4 \sin^2 \left(\frac{t}{2} \right). \end{aligned}$$

Hence the speed is

$$2 \left| \sin \left(\frac{t}{2} \right) \right|.$$

3. On $0 \leq t \leq 2\pi$, we have $0 \leq t/2 \leq \pi$, so $\sin(t/2) \geq 0$. Thus

$$L = 2 \int_0^{2\pi} \sin \left(\frac{t}{2} \right) dt.$$

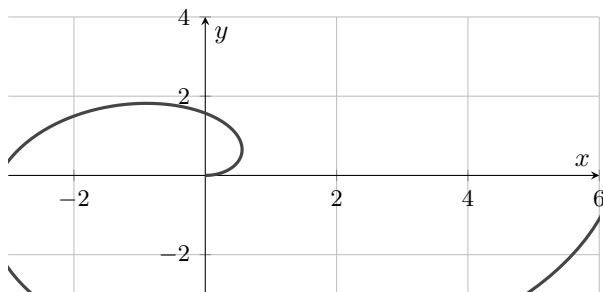
4. Evaluate the integral:

$$L = 2 \left[-2 \cos \left(\frac{t}{2} \right) \right]_0^{2\pi} = -4(\cos \pi - \cos 0) = \boxed{8}.$$

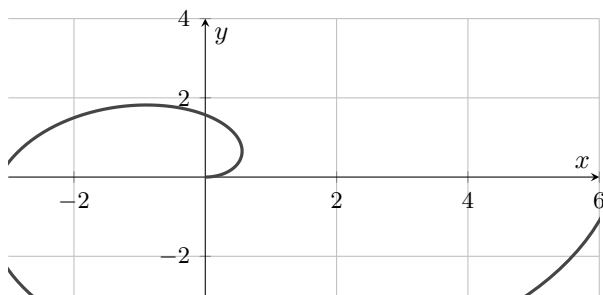
Question 3

Problem

A spiral-like path is $x = t \cos t$, $y = t \sin t$, $0 \leq t \leq 2\pi$. Set up its exact length and simplify the speed.



Question 3 – Solution



Solution

1. Apply the product rule to each coordinate:

$$x'(t) = \cos t - t \sin t, \quad y'(t) = \sin t + t \cos t.$$

2. Square and add:

$$\begin{aligned} x'^2 + y'^2 &= (\cos t - t \sin t)^2 + (\sin t + t \cos t)^2 \\ &= \cos^2 t + \sin^2 t + t^2(\sin^2 t + \cos^2 t) \\ &\quad - 2t \sin t \cos t + 2t \sin t \cos t \\ &= 1 + t^2. \end{aligned}$$

The cross terms cancel exactly.

3. Therefore, the speed is $\sqrt{1 + t^2}$, and the exact length is

$$L = \int_0^{2\pi} \sqrt{1 + t^2} \, dt.$$

4. If an evaluated form is desired, use

$$\int \sqrt{1 + t^2} \, dt = \frac{1}{2} \left(t\sqrt{1 + t^2} + \operatorname{arsinh} t \right),$$

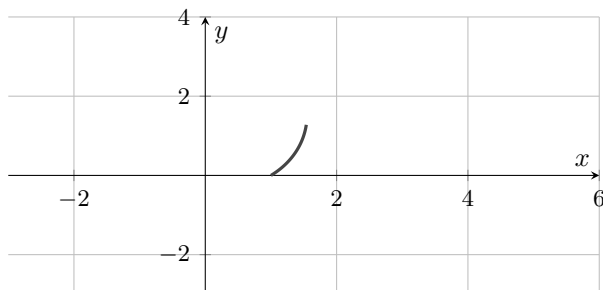
giving

$$L = \pi\sqrt{1 + 4\pi^2} + \frac{1}{2} \operatorname{arsinh}(2\pi).$$

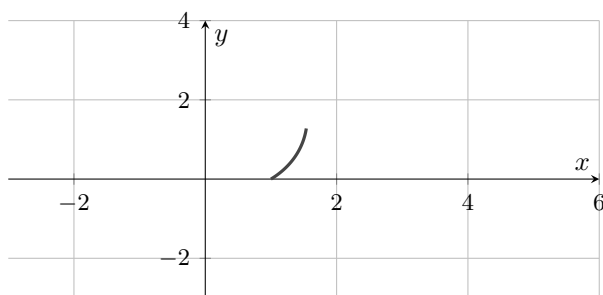
Question 4

Problem

Show that $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \ln 2$, has length $\sqrt{2}$.



Question 4 – Solution



Solution

1. Differentiate using the product rule:

$$x'(t) = e^t(\cos t - \sin t), \quad y'(t) = e^t(\sin t + \cos t).$$

2. Compute the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= e^{2t} [(\cos t - \sin t)^2 + (\sin t + \cos t)^2] \\ &= e^{2t} [2 \cos^2 t + 2 \sin^2 t] \\ &= 2e^{2t}. \end{aligned}$$

Since $e^t > 0$, the speed is $\sqrt{2}e^t$.

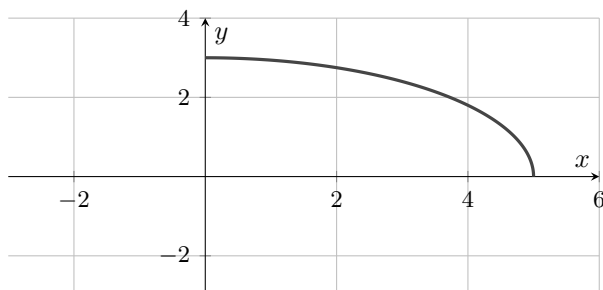
3. Set up and evaluate the length:

$$\begin{aligned} L &= \int_0^{\ln 2} \sqrt{2}e^t dt \\ &= \sqrt{2} [e^t]_0^{\ln 2} \\ &= \sqrt{2}(2 - 1) = \boxed{\sqrt{2}}. \end{aligned}$$

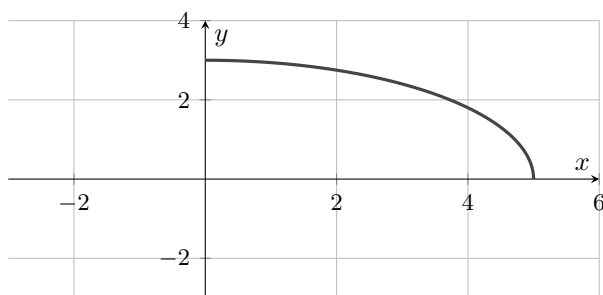
Question 5

Problem

A quarter ellipse is $x = 5 \cos t$, $y = 3 \sin t$, $0 \leq t \leq \pi/2$. Which simple quarter-circle lengths bound it?



Question 5 – Solution



Solution

1. Differentiate:

$$x'(t) = -5 \sin t, \quad y'(t) = 3 \cos t.$$

Thus the speed is

$$v(t) = \sqrt{25 \sin^2 t + 9 \cos^2 t}.$$

2. Rewrite the expression under the square root in two useful ways:

$$25 \sin^2 t + 9 \cos^2 t = 9 + 16 \sin^2 t = 25 - 16 \cos^2 t.$$

Since $0 \leq \sin^2 t, \cos^2 t \leq 1$, it follows that

$$9 \leq v(t)^2 \leq 25, \quad \text{so} \quad 3 \leq v(t) \leq 5.$$

3. Integrate these bounds over an interval of width $\pi/2$:

$$3 \int_0^{\pi/2} dt \leq L \leq 5 \int_0^{\pi/2} dt.$$

Therefore,

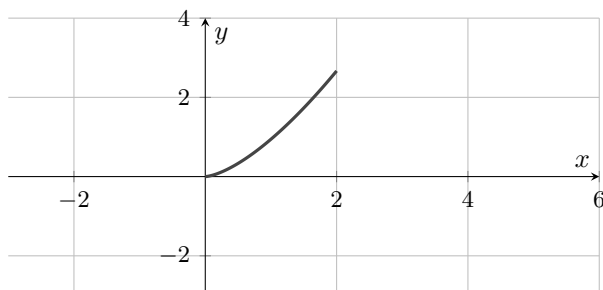
$$\boxed{\frac{3\pi}{2} \leq L \leq \frac{5\pi}{2}}.$$

These are the quarter-circumferences of circles with radii 3 and 5.

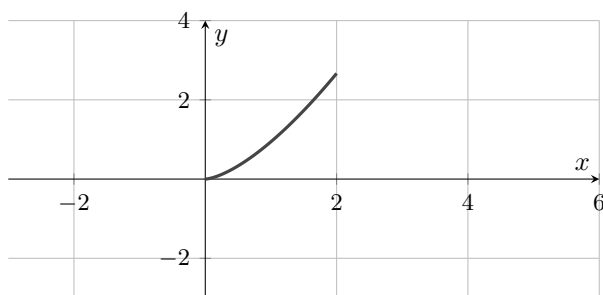
Question 6

Problem

Find the length of $x = t^2/2$, $y = t^3/3$, $0 \leq t \leq 2$.



Question 6 – Solution



Solution

1. Differentiate the coordinates:

$$x'(t) = t, \quad y'(t) = t^2.$$

2. The speed is

$$\sqrt{t^2 + t^4} = \sqrt{t^2(1 + t^2)} = |t|\sqrt{1 + t^2}.$$

Since $0 \leq t \leq 2$, $|t| = t$.

3. Therefore,

$$L = \int_0^2 t\sqrt{1 + t^2} dt.$$

Let $u = 1 + t^2$, so $du = 2t dt$. Then

$$\begin{aligned} L &= \frac{1}{2} \int_1^5 u^{1/2} du \\ &= \frac{1}{3} \left[u^{3/2} \right]_1^5 \\ &= \frac{1}{3} (5^{3/2} - 1). \end{aligned}$$

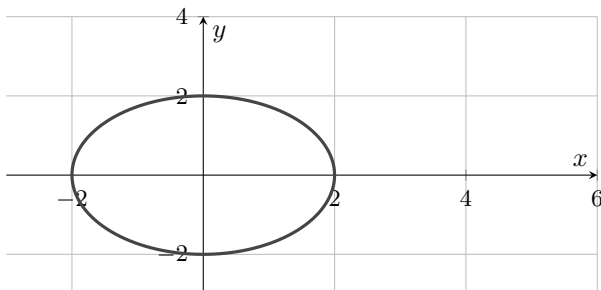
4. Hence

$$L = \frac{5\sqrt{5} - 1}{3}.$$

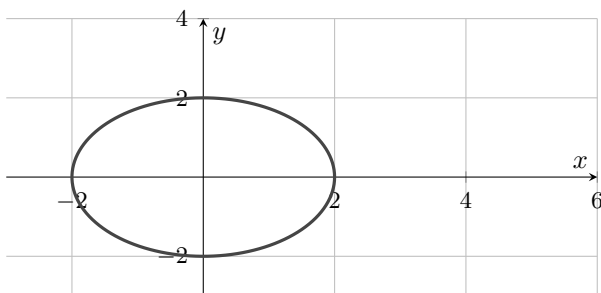
Question 7

Problem

A curve is reparametrized from t to $u = t^3$. Explain why its length is unchanged when both parameters increase.



Question 7 – Solution



Solution

1. Let the original curve be $\mathbf{r}(t)$ for $a \leq t \leq b$, and suppose $u = t^3$ increases on this interval. Then

$$t = u^{1/3}, \quad \frac{dt}{du} = \frac{1}{3u^{2/3}}$$

wherever this derivative is defined.

2. By the chain rule, the velocity in the new parameter is

$$\frac{d\mathbf{r}}{du} = \frac{d\mathbf{r}}{dt} \frac{dt}{du}.$$

Taking magnitudes gives

$$\left\| \frac{d\mathbf{r}}{du} \right\| = \left\| \frac{d\mathbf{r}}{dt} \right\| \left| \frac{dt}{du} \right|.$$

3. Because both parameters increase, $dt/du \geq 0$ and $dt = (dt/du)du$. Hence

$$\left\| \frac{d\mathbf{r}}{du} \right\| du = \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$$

Integrating over corresponding endpoints yields the same value in either parameter.

4. Therefore, reparametrization changes the rate at which the curve is traced but not the geometric arc length:

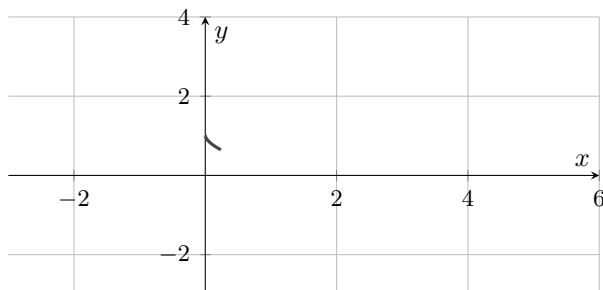
$$\boxed{L_u = L_t}.$$

A possible derivative singularity at the single point $u = 0$ does not change the value of the proper or corresponding improper integral.

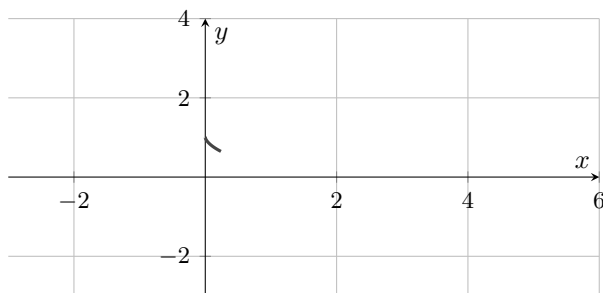
Question 8

Problem

Find the length of the tractable curve $x = t - \tanh t$, $y = \operatorname{sech} t$, $0 \leq t \leq 1$.



Question 8 – Solution



Solution

1. Differentiate and use $1 - \operatorname{sech}^2 t = \tanh^2 t$:

$$x'(t) = 1 - \operatorname{sech}^2 t = \tanh^2 t,$$

$$y'(t) = -\operatorname{sech} t \tanh t.$$

2. Simplify the squared speed:

$$\begin{aligned} x'^2 + y'^2 &= \tanh^4 t + \operatorname{sech}^2 t \tanh^2 t \\ &= \tanh^2 t (\tanh^2 t + \operatorname{sech}^2 t) \\ &= \tanh^2 t. \end{aligned}$$

Thus the speed is $|\tanh t|$.

3. On $0 \leq t \leq 1$, $\tanh t \geq 0$, so

$$L = \int_0^1 \tanh t \, dt.$$

Since $d(\ln \cosh t)/dt = \tanh t$,

$$L = [\ln(\cosh t)]_0^1 = \ln(\cosh 1) - \ln 1.$$

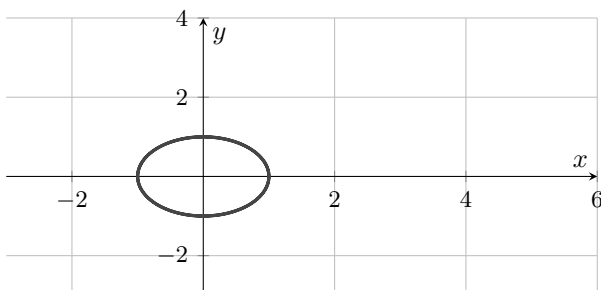
4. Therefore,

$$\boxed{L = \ln(\cosh 1)}.$$

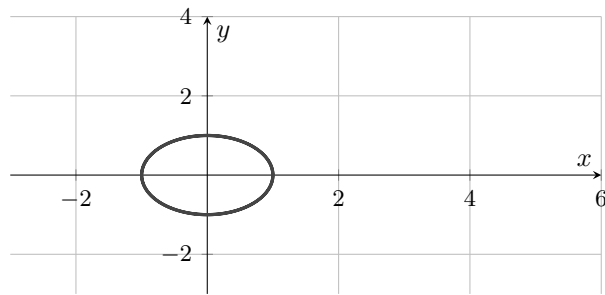
Question 9

Problem

A particle goes around the unit circle twice. Is its path length 2π or 4π ? Justify using $x = \cos 2t$, $y = \sin 2t$, $0 \leq t \leq 2\pi$.



Question 9 – Solution



Solution

1. Differentiate:

$$x'(t) = -2 \sin 2t, \quad y'(t) = 2 \cos 2t.$$

2. The speed is constant:

$$\sqrt{x'^2 + y'^2} = \sqrt{4 \sin^2 2t + 4 \cos^2 2t} = 2.$$

3. Integrate over the full parameter interval:

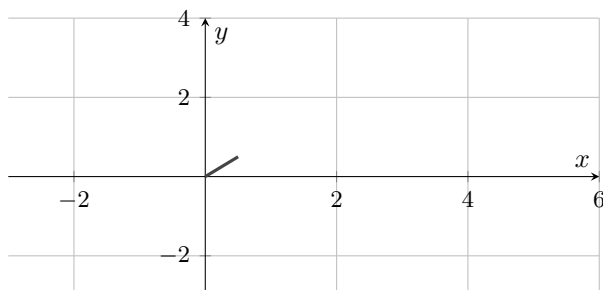
$$L = \int_0^{2\pi} 2 \, dt = \boxed{4\pi}.$$

4. The angle of the point is $2t$, which increases from 0 to 4π . Therefore, the unit circle is traversed twice. Its geometric circumference is 2π , but the particle travels that circumference twice, producing total traveled length 4π .

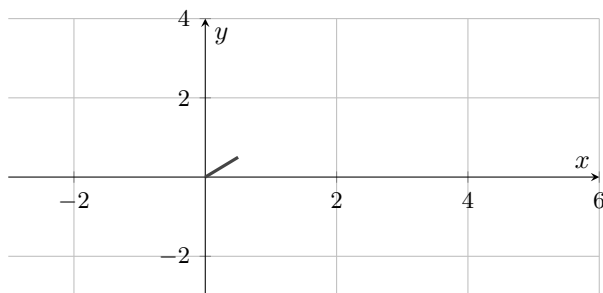
Question 10

Problem

Find $a > 0$ so that the line path $x = at$, $y = (1 - a)t$, $0 \leq t \leq 1$, has minimum length.



Question 10 – Solution



Solution

1. Differentiate the line parametrization:

$$x'(t) = a, \quad y'(t) = 1 - a.$$

The speed is constant, so

$$L(a) = \int_0^1 \sqrt{a^2 + (1-a)^2} dt = \sqrt{a^2 + (1-a)^2}.$$

2. Because the square-root function is increasing, minimizing $L(a)$ is equivalent to minimizing

$$L(a)^2 = a^2 + (1-a)^2 = 2a^2 - 2a + 1.$$

Complete the square:

$$2a^2 - 2a + 1 = 2 \left(a - \frac{1}{2} \right)^2 + \frac{1}{2}.$$

3. The squared term is smallest when $a = 1/2$, which satisfies $a > 0$. Therefore,

$$\boxed{a = \frac{1}{2}}.$$

4. At this value, the minimum length is

$$L_{\min} = \sqrt{\frac{1}{2}} = \boxed{\frac{1}{\sqrt{2}}}.$$

Geometrically, the endpoint $(a, 1-a)$ is then $(1/2, 1/2)$, the closest point on the line $x + y = 1$ to the origin.