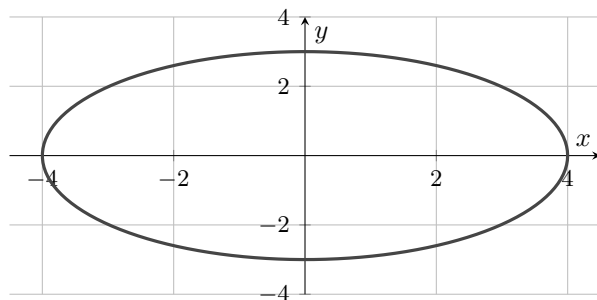
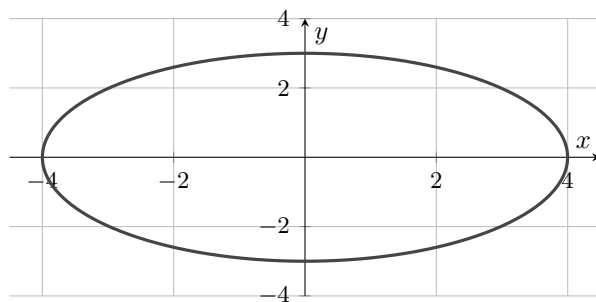


Question 1**Problem**

Use a parametric integral to find the area of the ellipse $x = 4 \cos t$, $y = 3 \sin t$.



Question 1 – Solution



Solution

1. The interval $0 \leq t \leq \pi/2$ traces the ellipse's first-quadrant arc from $(4, 0)$ to $(0, 3)$. By symmetry, the total area is four times the area in that quadrant.

2. Differentiate the horizontal coordinate:

$$\frac{dx}{dt} = -4 \sin t.$$

Because x decreases on this interval, the first-quadrant area is

$$A_1 = - \int_0^{\pi/2} y(t)x'(t) dt.$$

3. Substitute $y = 3 \sin t$ and $x' = -4 \sin t$:

$$A_1 = - \int_0^{\pi/2} (3 \sin t)(-4 \sin t) dt = 12 \int_0^{\pi/2} \sin^2 t dt.$$

4. Using $\sin^2 t = (1 - \cos 2t)/2$,

$$\int_0^{\pi/2} \sin^2 t dt = \frac{\pi}{4}.$$

Hence $A_1 = 3\pi$.

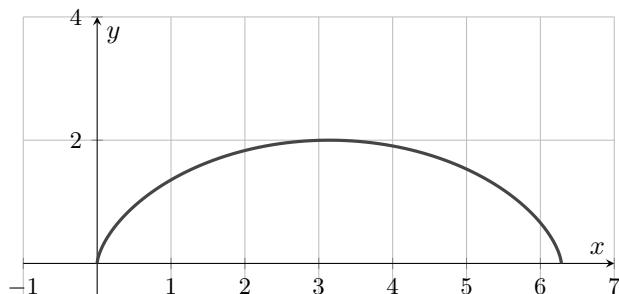
5. Multiply by four quadrants:

$$A = 4A_1 = \boxed{12\pi}.$$

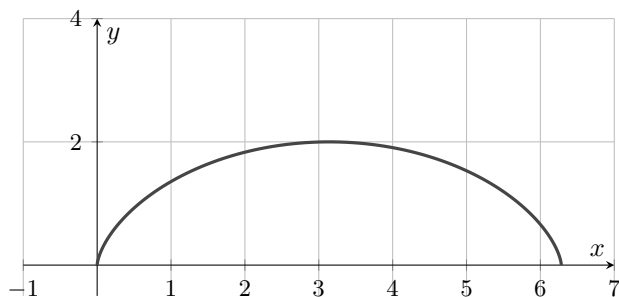
Question 2

Problem

Find the area under one cycloid arch $x = t - \sin t$, $y = 1 - \cos t$, $0 \leq t \leq 2\pi$.



Question 2 – Solution



Solution

1. The given interval traces one arch from $(0,0)$ to $(2\pi,0)$. Since

$$x'(t) = 1 - \cos t \geq 0,$$

the curve moves from left to right, so the area under it is $\int y \, dx$ with the stated bounds.

2. Substitute $y(t) = 1 - \cos t$ and $dx = x'(t)dt$:

$$A = \int_0^{2\pi} y(t)x'(t) \, dt = \int_0^{2\pi} (1 - \cos t)^2 \, dt.$$

3. Expand the integrand:

$$(1 - \cos t)^2 = 1 - 2\cos t + \cos^2 t.$$

Therefore,

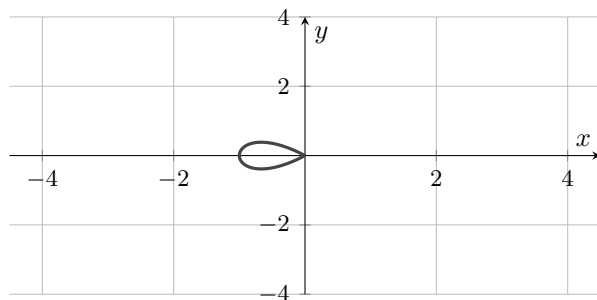
$$A = \int_0^{2\pi} 1 \, dt - 2 \int_0^{2\pi} \cos t \, dt + \int_0^{2\pi} \cos^2 t \, dt.$$

4. These three integrals equal 2π , 0, and π , respectively. Thus

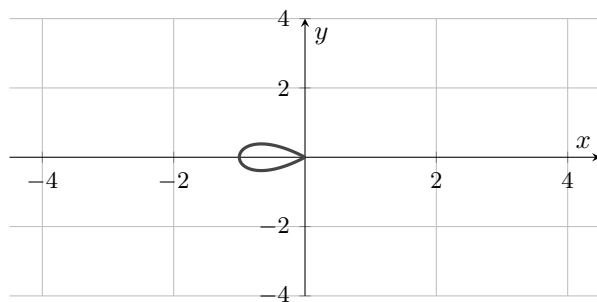
$$\boxed{A = 3\pi}.$$

Question 3**Problem**

A loop is traced by $x = t^2 - 1$, $y = t^3 - t$. Find the exact area enclosed by the loop.



Question 3 – Solution

**Solution**

1. Locate the parameter values at which the loop closes. Since

$$x = t^2 - 1, \quad y = t(t^2 - 1),$$

both $t = -1$ and $t = 1$ produce the point $(0, 0)$. The interval $-1 \leq t \leq 1$ traces the loop once.

2. Differentiate x :

$$x'(t) = 2t.$$

The signed area integral is

$$\int_{-1}^1 y(t)x'(t) dt = \int_{-1}^1 (t^3 - t)(2t) dt.$$

3. Simplify and integrate:

$$\begin{aligned} \int_{-1}^1 (2t^4 - 2t^2) dt &= \left[\frac{2}{5}t^5 - \frac{2}{3}t^3 \right]_{-1}^1 \\ &= \frac{4}{5} - \frac{4}{3} = -\frac{8}{15}. \end{aligned}$$

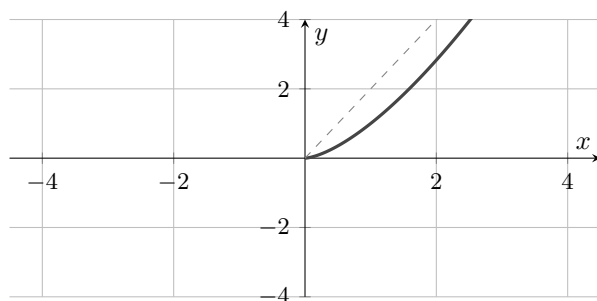
4. The negative sign records the orientation of traversal; geometric area is nonnegative. Hence

$$A = \left| -\frac{8}{15} \right| = \frac{8}{15}.$$

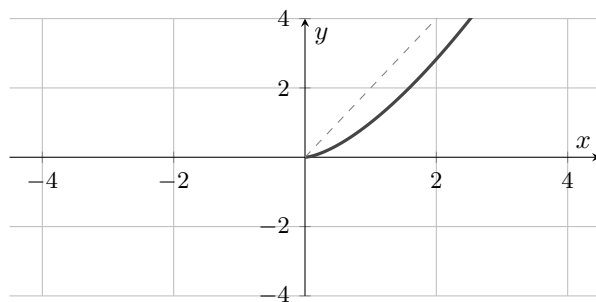
Question 4

Problem

The curve $x = t^2$, $y = t^3$ joins $(0,0)$ to $(4,8)$. Find the area between it and the chord joining those endpoints.



Question 4 – Solution



Solution

1. The endpoints correspond to $t = 0$ and $t = 2$. The chord through $(0, 0)$ and $(4, 8)$ has slope

$$m = \frac{8 - 0}{4 - 0} = 2,$$

so its equation is $y = 2x$.

2. Along the parametric curve, $x = t^2$ and $y = t^3$. The chord's height at the same x -coordinate is

$$y_{\text{chord}} = 2x = 2t^2.$$

For $0 \leq t \leq 2$, $2t^2 - t^3 = t^2(2 - t) \geq 0$, so the chord lies above the curve.

3. Since $dx/dt = 2t$, the area between them is

$$A = \int_0^2 (y_{\text{chord}} - y_{\text{curve}}) dx = \int_0^2 (2t^2 - t^3)(2t) dt.$$

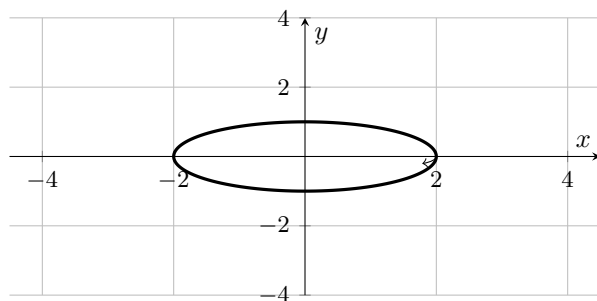
4. Evaluate:

$$\begin{aligned} A &= \int_0^2 (4t^3 - 2t^4) dt \\ &= \left[t^4 - \frac{2}{5}t^5 \right]_0^2 = 16 - \frac{64}{5} = \boxed{\frac{16}{5}}. \end{aligned}$$

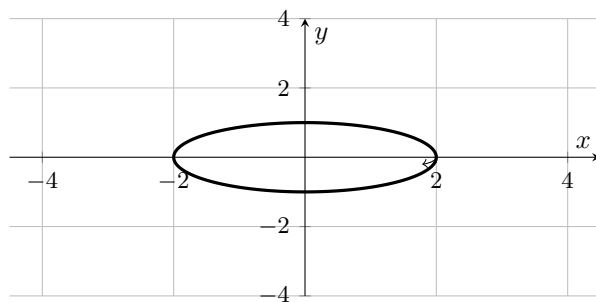
Question 5

Problem

A simple, piecewise smooth closed curve is traversed once clockwise. Explain why $\int y \, dx$ may be positive even though area cannot be negative.



Question 5 – Solution



Solution

1. A line integral around a closed curve is signed: reversing the direction of traversal reverses the sign because dx changes sign.
2. For a positively oriented, counterclockwise curve C , Green's theorem gives

$$A = - \oint_C y \, dx.$$

Thus $\oint_C y \, dx = -A$ for counterclockwise traversal.

3. If the same boundary is traversed clockwise, the line integral changes sign:

$$\oint_{-C} y \, dx = - \oint_C y \, dx = A.$$

Therefore, $\int y \, dx$ can be positive for clockwise motion; the sign describes orientation, not a negative or positive physical area by itself.

4. A direction-independent formula for geometric area is

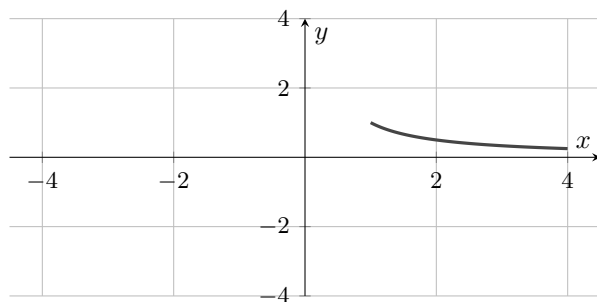
$$A = \left| \oint_C y \, dx \right|.$$

Equivalently, use $-\oint_C y \, dx$ with counterclockwise orientation or reverse the bounds when necessary.

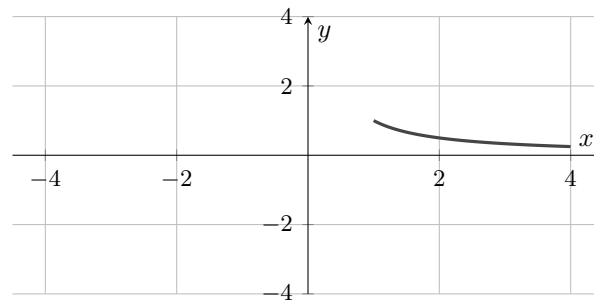
Question 6

Problem

Find the area swept under $x = e^t$, $y = e^{-t}$ from $t = 0$ to $t = \ln 4$.



Question 6 – Solution



Solution

1. On $0 \leq t \leq \ln 4$, $x = e^t$ increases from 1 to 4 because

$$x'(t) = e^t > 0.$$

Therefore, the requested area under the curve is directly $\int y \, dx$.

2. Substitute the parametric expressions:

$$A = \int_0^{\ln 4} y(t)x'(t) \, dt = \int_0^{\ln 4} e^{-t} e^t \, dt.$$

3. The exponential factors cancel:

$$e^{-t} e^t = 1.$$

Hence

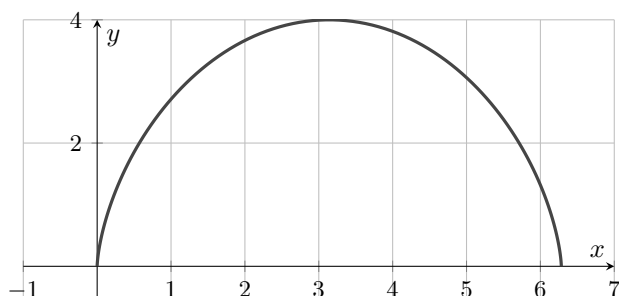
$$A = \int_0^{\ln 4} 1 \, dt = [t]_0^{\ln 4} = \boxed{\ln 4}.$$

4. As a check, eliminating t gives $y = 1/x$, and $\int_1^4 (1/x) \, dx = \ln 4$, the same result.

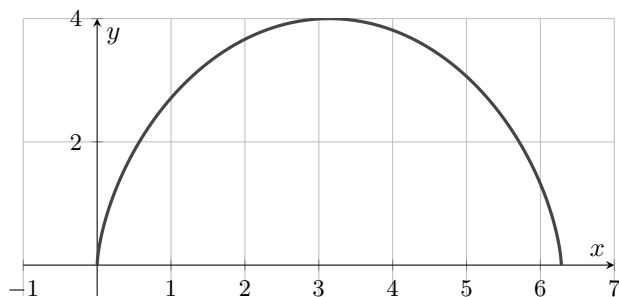
Question 7

Problem

The parametric curve $x = t - \sin t$, $y = 2 - 2 \cos t$ is a stretched cycloid. Predict, then compute, how its arch area compares with the standard cycloid.



Question 7 – Solution



Solution

1. Compared with the standard cycloid $x = t - \sin t$, $y = 1 - \cos t$, the x -coordinates are unchanged while every y -coordinate is multiplied by 2. A vertical stretch by a factor of 2 doubles area, so the predicted arch area is $2(3\pi) = 6\pi$.
2. Verify this prediction directly on $0 \leq t \leq 2\pi$. Differentiate x :

$$x'(t) = 1 - \cos t.$$

Since $x'(t) \geq 0$, the arch is traced from left to right.

3. Substitute $y = 2(1 - \cos t)$:

$$A = \int_0^{2\pi} y(t)x'(t) dt = 2 \int_0^{2\pi} (1 - \cos t)^2 dt.$$

4. From

$$\int_0^{2\pi} (1 - \cos t)^2 dt = 2\pi + \pi = 3\pi,$$

it follows that

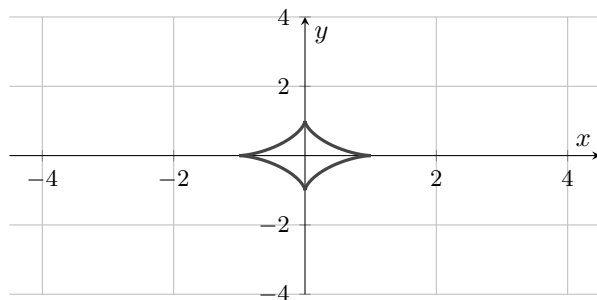
$$\boxed{A = 2(3\pi) = 6\pi}.$$

The computation agrees with the scaling prediction.

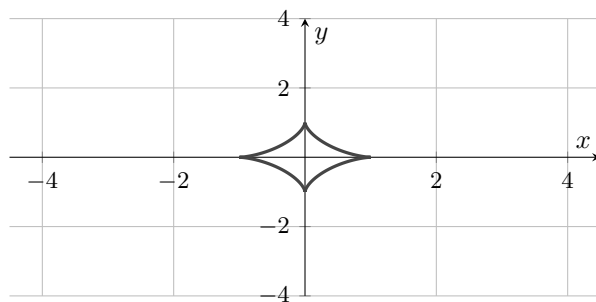
Question 8

Problem

Find the area enclosed by $x = \cos^3 t$, $y = \sin^3 t$ without first eliminating the parameter.



Question 8 – Solution

**Solution**

1. The interval $0 \leq t \leq \pi/2$ traces the first-quadrant arc from $(1, 0)$ to $(0, 1)$. By symmetry, the enclosed area is four times the first-quadrant area.
2. It is convenient to integrate with respect to y :

$$A_1 = \int x \, dy = \int_0^{\pi/2} x(t) y'(t) \, dt.$$

Since $y = \sin^3 t$,

$$y'(t) = 3 \sin^2 t \cos t.$$

3. Substitute $x = \cos^3 t$:

$$A_1 = 3 \int_0^{\pi/2} \sin^2 t \cos^4 t \, dt.$$

Using the even-power integral (or power-reduction identities),

$$\int_0^{\pi/2} \sin^2 t \cos^4 t \, dt = \frac{\pi}{32}.$$

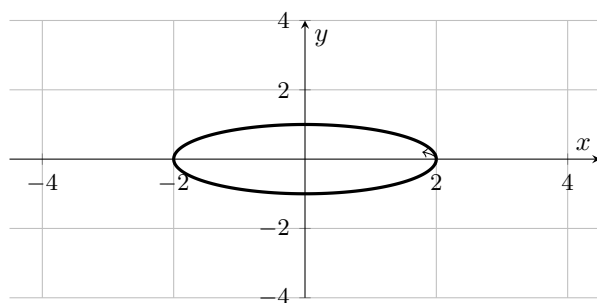
Thus $A_1 = 3\pi/32$.

4. Multiply by four quadrants:

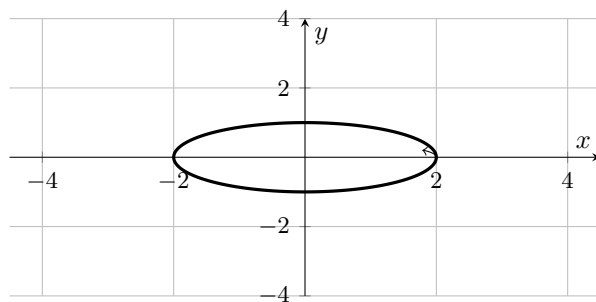
$$A = 4A_1 = 4 \left(\frac{3\pi}{32} \right) = \boxed{\frac{3\pi}{8}}.$$

Question 9**Problem**

A designer reverses a parametrization on $0 \leq t \leq 1$ by replacing t with $1 - t$. What happens to the signed area integral $\int yx' dt$?



Question 9 – Solution

**Solution**

1. Let the original parametrization be $(x(t), y(t))$ on $0 \leq t \leq 1$, with signed integral

$$I = \int_0^1 y(t)x'(t) dt.$$

2. Under the reversed parametrization, set $u = 1 - t$. The new coordinates are

$$\tilde{x}(t) = x(1 - t), \quad \tilde{y}(t) = y(1 - t).$$

By the chain rule,

$$\tilde{x}'(t) = -x'(1 - t).$$

3. The reversed signed integral is

$$\begin{aligned} \tilde{I} &= \int_0^1 \tilde{y}(t)\tilde{x}'(t) dt \\ &= - \int_0^1 y(1 - t)x'(1 - t) dt. \end{aligned}$$

Substituting $u = 1 - t$ shows that $\tilde{I} = -I$.

4. Therefore, reversing the parametrization changes the sign of the signed area integral but not its magnitude:

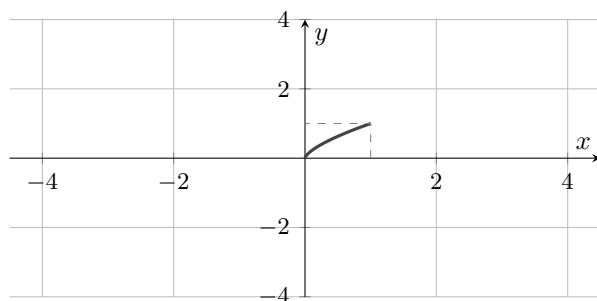
$$\boxed{\tilde{I} = -I, \quad |\tilde{I}| = |I|}.$$

The geometric area is unchanged.

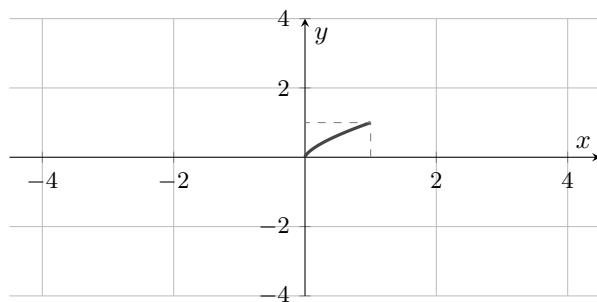
Question 10

Problem

For $x = t^3$, $y = t^2$, $0 \leq t \leq 1$, compare the area under the curve with the area to its left inside the unit square.



Question 10 – Solution

**Solution**

1. As t increases from 0 to 1, the curve runs from $(0, 0)$ to $(1, 1)$ with

$$x'(t) = 3t^2 \geq 0.$$

The area below the curve is therefore

$$A_{\text{below}} = \int_0^1 y(t)x'(t) dt.$$

2. Substitute $y = t^2$ and $x' = 3t^2$:

$$A_{\text{below}} = \int_0^1 3t^4 dt = \left[\frac{3}{5}t^5 \right]_0^1 = \boxed{\frac{3}{5}}.$$

3. The curve divides the unit square into the area below it and the area to its left (equivalently, above it). Since the square has area 1,

$$A_{\text{left}} = 1 - A_{\text{below}} = 1 - \frac{3}{5} = \boxed{\frac{2}{5}}.$$

4. This can also be checked directly with $dy = 2t dt$:

$$A_{\text{left}} = \int_0^1 x dy = \int_0^1 t^3(2t) dt = \frac{2}{5}.$$

Thus the area under the curve is larger by $1/5$.