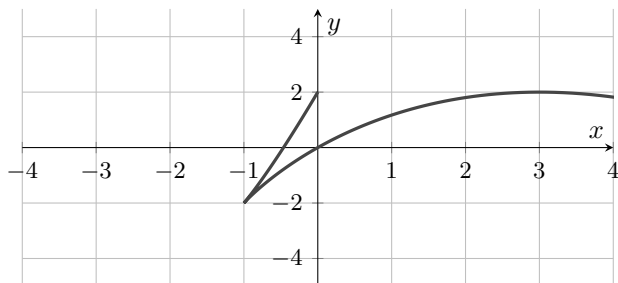


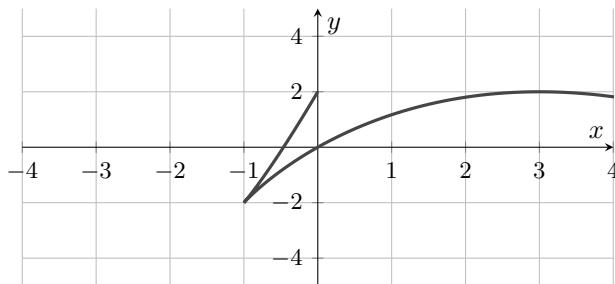
Question 1

Problem

For $x = t^2 - 2t$, $y = t^3 - 3t$, find every horizontal and vertical tangent.



Question 1 – Solution



Solution

1. Differentiate each coordinate:

$$\frac{dx}{dt} = 2t - 2 = 2(t - 1), \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t - 1)(t + 1).$$

2. A horizontal tangent requires $dy/dt = 0$ while $dx/dt \neq 0$. The candidates are

$$3(t - 1)(t + 1) = 0 \implies t = -1, 1.$$

At $t = -1$, $dx/dt = -4 \neq 0$, so this value gives a horizontal tangent. The point is

$$x(-1) = 3, \quad y(-1) = 2.$$

Thus the horizontal tangent line is

$$\boxed{y = 2}.$$

3. At $t = 1$, both derivatives are zero, so the usual horizontal- or vertical-tangent tests are inconclusive. For $t \neq 1$, cancel the common factor:

$$\frac{dy}{dx} = \frac{3(t - 1)(t + 1)}{2(t - 1)} = \frac{3}{2}(t + 1).$$

Therefore,

$$\lim_{t \rightarrow 1} \frac{dy}{dx} = 3.$$

The point at $t = 1$ is $(-1, -2)$, and its tangent line is $y + 2 = 3(x + 1)$; it is neither horizontal nor vertical.

4. A vertical tangent would require $dx/dt = 0$ with $dy/dt \neq 0$. The only zero of dx/dt is $t = 1$, where dy/dt is also zero and the limiting slope is finite. Hence there are no vertical tangents.

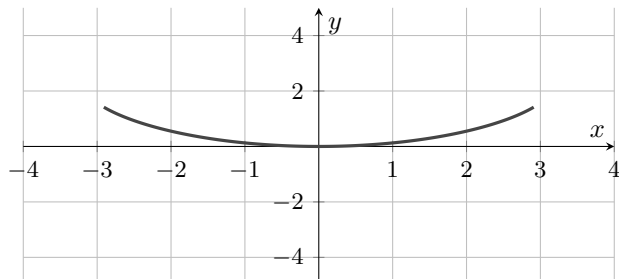
5. The complete answer is

$$\boxed{\text{horizontal tangent } y = 2 \text{ at } (3, 2); \quad \text{no vertical tangents.}}$$

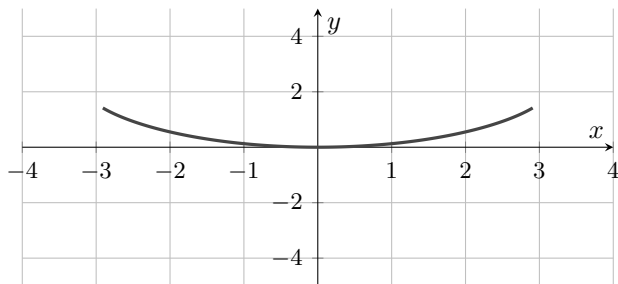
Question 2

Problem

The path $x = t + \sin t$, $y = 1 - \cos t$ models a rolling marker. Find its tangent slope at $t = \pi/2$ and interpret its sign.



Question 2 – Solution



Solution

1. Differentiate the coordinate functions:

$$\frac{dx}{dt} = 1 + \cos t, \quad \frac{dy}{dt} = \sin t.$$

2. At $t = \pi/2$,

$$\frac{dx}{dt} = 1 + \cos \frac{\pi}{2} = 1, \quad \frac{dy}{dt} = \sin \frac{\pi}{2} = 1.$$

Since $dx/dt \neq 0$, the tangent slope is

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{1} = \boxed{1}.$$

3. The point on the path is

$$\left(x\left(\frac{\pi}{2}\right), y\left(\frac{\pi}{2}\right)\right) = \left(\frac{\pi}{2} + 1, 1\right).$$

Therefore, the tangent line is

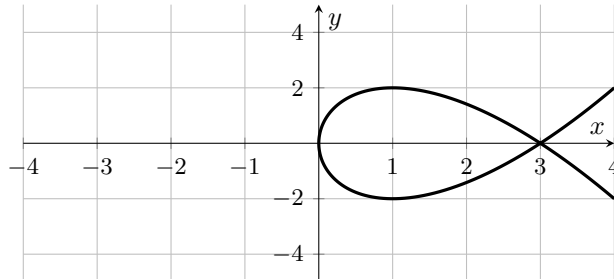
$$y - 1 = x - \left(\frac{\pi}{2} + 1\right).$$

4. Both velocity components are positive at this instant. Thus the marker is moving to the right and upward, which is consistent with the positive slope.

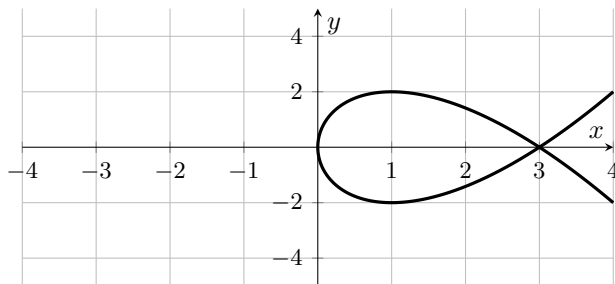
Question 3

Problem

At which points does $x = t^2$, $y = t^3 - 3t$ have tangent slope 0? Is the particle momentarily stopped there?



Question 3 – Solution



Solution

1. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 3 = 3(t-1)(t+1).$$

2. A tangent has slope 0 when $dy/dt = 0$ and $dx/dt \neq 0$. Solve

$$3(t-1)(t+1) = 0,$$

which gives $t = -1$ and $t = 1$.

3. Check the horizontal velocity:

$$x'(-1) = -2 \neq 0, \quad x'(1) = 2 \neq 0.$$

Thus both parameter values give genuine horizontal tangents.

4. Find the corresponding points:

$$\mathbf{r}(-1) = (1, 2), \quad \mathbf{r}(1) = (1, -2).$$

The horizontal tangent lines are therefore $y = 2$ and $y = -2$.

5. A particle is momentarily stopped only when both velocity components are zero. At $t = \pm 1$, $dy/dt = 0$ but $dx/dt = \pm 2$ is nonzero, so the particle is moving horizontally rather than stopping.

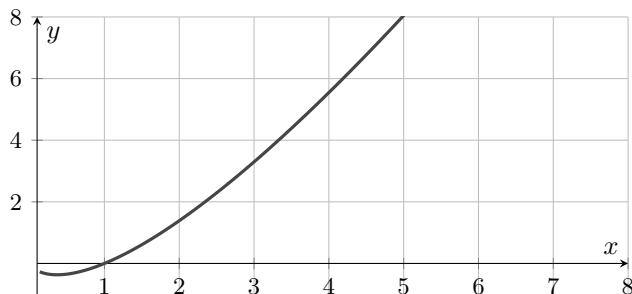
6. Hence the requested points are

$$\boxed{(1, 2) \text{ and } (1, -2)},$$

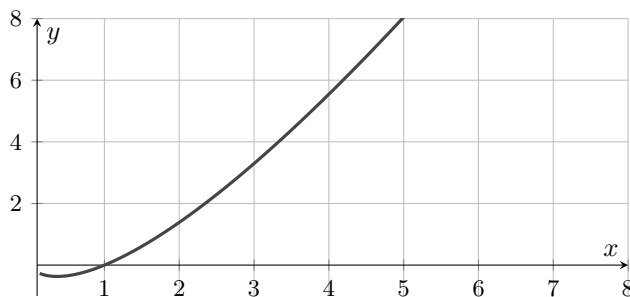
and the particle is not momentarily stopped at either point.

Question 4**Problem**

For $x = e^t$, $y = te^t$, find the tangent line at the point whose x -coordinate is 1.



Question 4 – Solution



Solution

1. Locate the required parameter value by solving

$$x = e^t = 1.$$

Since the exponential function equals 1 only at $t = 0$, the required parameter value is $t = 0$.

2. Evaluate the y -coordinate:

$$y(0) = 0 \cdot e^0 = 0.$$

Thus the point of tangency is $(1, 0)$.

3. Differentiate both coordinates. The product rule is needed for y :

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = e^t + te^t = e^t(1 + t).$$

4. Because $dx/dt = e^t \neq 0$, divide the derivatives:

$$\frac{dy}{dx} = \frac{e^t(1 + t)}{e^t} = 1 + t.$$

At $t = 0$, the slope is $m = 1$.

5. Use point-slope form through $(1, 0)$:

$$y - 0 = 1(x - 1).$$

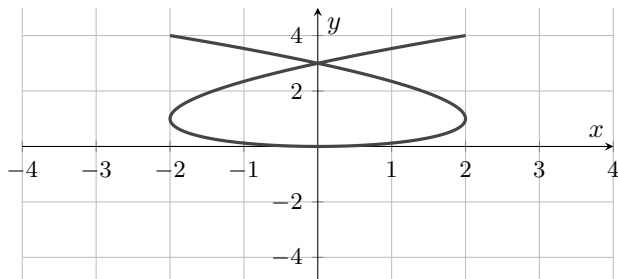
Therefore, the tangent line is

$$\boxed{y = x - 1}.$$

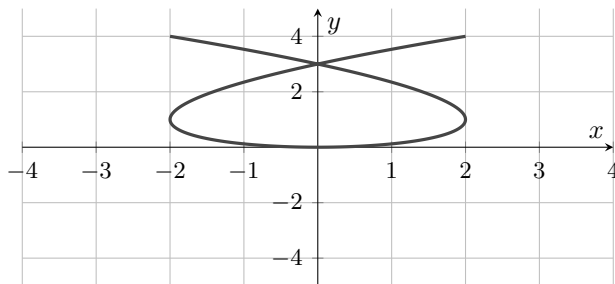
Question 5

Problem

A student claims that the two tangents to $x = t^3 - 3t$, $y = t^2$ at its self-intersection are perpendicular. Locate the crossing and test the claim.



Question 5 – Solution



Solution

- At a self-intersection, two distinct parameter values give the same y -coordinate. Since $y = t^2$, equal y -coordinates occur for parameters t and $-t$.
- Require the corresponding x -coordinates to be equal:

$$t^3 - 3t = (-t)^3 - 3(-t) = -t^3 + 3t.$$

Simplifying gives

$$2t^3 - 6t = 2t(t^2 - 3) = 0.$$

The value $t = 0$ does not give two distinct parameters. The nontrivial pair is

$$t = \pm\sqrt{3}.$$

- At either parameter value,

$$x = t(t^2 - 3) = 0, \quad y = t^2 = 3.$$

Hence the self-intersection is $\boxed{(0, 3)}$.

- Differentiate:

$$\frac{dx}{dt} = 3t^2 - 3, \quad \frac{dy}{dt} = 2t.$$

At $t = \pm\sqrt{3}$, $dx/dt = 6 \neq 0$, so

$$\frac{dy}{dx} = \frac{2t}{3t^2 - 3} = \frac{t}{3}.$$

The two slopes are

$$m_1 = \frac{1}{\sqrt{3}}, \quad m_2 = -\frac{1}{\sqrt{3}}.$$

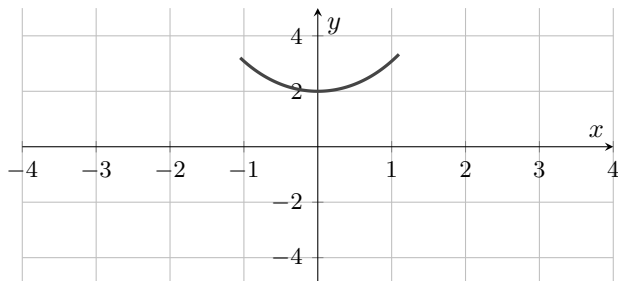
- Two nonvertical lines are perpendicular only if the product of their slopes is -1 . Here

$$m_1 m_2 = -\frac{1}{3} \neq -1.$$

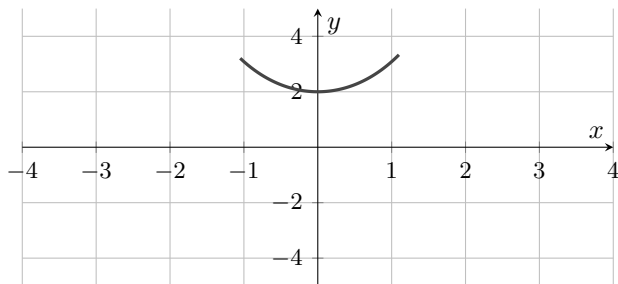
Therefore, the two tangents are $\boxed{\text{not perpendicular}}$.

Question 6**Problem**

For $x = \ln t$, $y = t + t^{-1}$, find d^2y/dx^2 at $t = 1$ and classify the local shape.



Question 6 – Solution



Solution

1. Because $x = \ln t$, the parameter domain is $t > 0$. Differentiate the coordinates:

$$\frac{dx}{dt} = \frac{1}{t}, \quad \frac{dy}{dt} = 1 - \frac{1}{t^2}.$$

2. Compute the first derivative with respect to x :

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - t^{-2}}{t^{-1}} = t - \frac{1}{t}.$$

3. For a parametric curve, differentiate dy/dx with respect to t and divide once more by dx/dt :

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{dx/dt}.$$

Since

$$\frac{d}{dt} \left(t - \frac{1}{t} \right) = 1 + \frac{1}{t^2},$$

we obtain

$$\frac{d^2y}{dx^2} = \frac{1 + t^{-2}}{t^{-1}} = t + \frac{1}{t}.$$

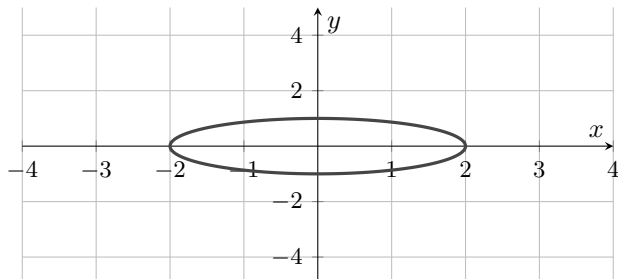
4. Evaluate at $t = 1$:

$$\left. \frac{d^2y}{dx^2} \right|_{t=1} = 1 + 1 = \boxed{2}.$$

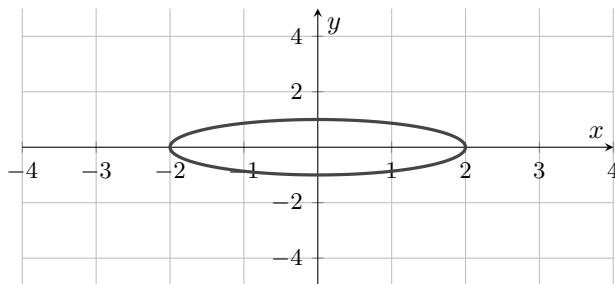
5. Because the second derivative is positive, the curve is concave up at this point. Also, $(x(1), y(1)) = (0, 2)$ and $dy/dx = 0$, so $(0, 2)$ is a local minimum of the curve when viewed as y versus x .

Question 7**Problem**

Find the angle at which the parametric curve $x = 2 \cos t$, $y = \sin t$ crosses the line $y = x/2$ in the first quadrant.



Question 7 – Solution



Solution

1. At an intersection with $y = x/2$, substitute the parametric equations:

$$\sin t = \frac{2 \cos t}{2} = \cos t.$$

In the first quadrant, this gives $t = \pi/4$.

2. The intersection point is

$$\left(2 \cos \frac{\pi}{4}, \sin \frac{\pi}{4}\right) = \left(\sqrt{2}, \frac{\sqrt{2}}{2}\right).$$

3. Differentiate the curve:

$$\frac{dx}{dt} = -2 \sin t, \quad \frac{dy}{dt} = \cos t.$$

Thus its tangent slope at $t = \pi/4$ is

$$m_c = \frac{\cos t}{-2 \sin t} = -\frac{1}{2}.$$

The line $y = x/2$ has slope $m_\ell = 1/2$.

4. If ϕ is the acute angle between two lines, then

$$\tan \phi = \left| \frac{m_\ell - m_c}{1 + m_\ell m_c} \right|.$$

Therefore,

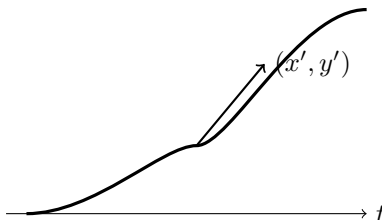
$$\tan \phi = \left| \frac{\frac{1}{2} - (-\frac{1}{2})}{1 + (\frac{1}{2})(-\frac{1}{2})} \right| = \frac{1}{3/4} = \frac{4}{3}.$$

5. Hence the crossing angle is

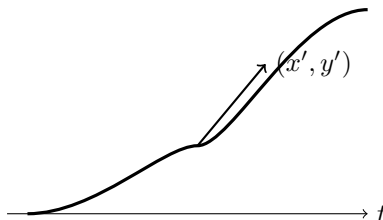
$$\phi = \arctan \left(\frac{4}{3} \right) \approx 53.13^\circ.$$

Question 8**Problem**

A particle has velocity $(x', y') = (t - 2, t^2 - 4)$. Does its path have a vertical tangent? Analyze the tangent and direction of motion at any stationary time.



Question 8 – Solution

**Solution**

1. A regular vertical tangent would require

$$x'(t) = t - 2 = 0 \quad \text{and} \quad y'(t) = t^2 - 4 \neq 0.$$

The first condition gives the only candidate, $t = 2$.

2. At $t = 2$,

$$(x'(2), y'(2)) = (0, 0).$$

Thus the particle is momentarily stopped, and the ordinary vertical-tangent test does not apply.

3. For $t \neq 2$, factor y' and simplify the slope:

$$\frac{dy}{dx} = \frac{t^2 - 4}{t - 2} = \frac{(t - 2)(t + 2)}{t - 2} = t + 2.$$

Consequently,

$$\lim_{t \rightarrow 2} \frac{dy}{dx} = 4,$$

which is finite rather than infinite. The path has a tangent of slope 4 at the stationary point, not a vertical tangent.

4. The direction also changes across $t = 2$. Just before 2, both x' and y' are negative, so the particle moves down and left. Just after 2, both are positive, so it moves up and right. At $t = 2$ itself, the velocity is zero and therefore has no instantaneous direction.

5. Hence

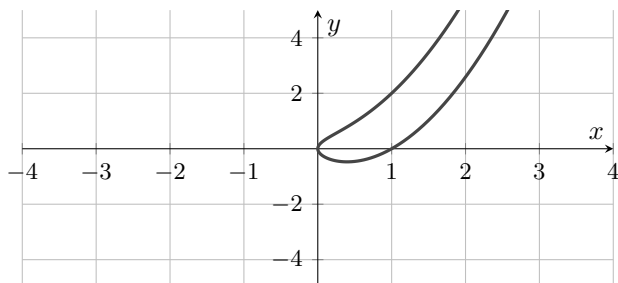
the path has no vertical tangent.

The only candidate is a stationary point where the particle reverses direction along a tangent of slope 4.

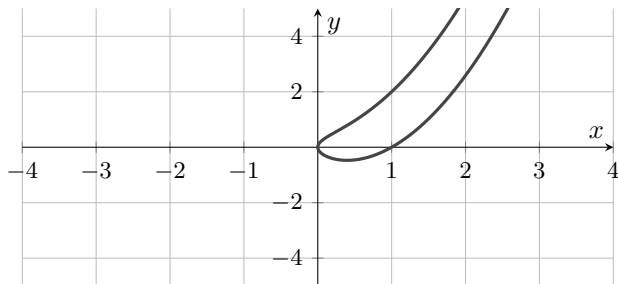
Question 9

Problem

The curve $x = t^2$, $y = t^4 - t$ passes through $(1, 0)$. Find all tangent lines there.



Question 9 – Solution



Solution

1. First find every parameter value that could produce the point $(1, 0)$. From $x = t^2 = 1$,

$$t = 1 \quad \text{or} \quad t = -1.$$

2. Check the y -coordinate for both values:

$$y(1) = 1^4 - 1 = 0, \quad y(-1) = (-1)^4 - (-1) = 2.$$

Only $t = 1$ produces $(1, 0)$, so there can be only one tangent line there.

3. Differentiate:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 4t^3 - 1.$$

At $t = 1$, $dx/dt = 2 \neq 0$, and the slope is

$$\frac{dy}{dx} = \frac{4t^3 - 1}{2t} = \frac{4(1)^3 - 1}{2(1)} = \frac{3}{2}.$$

4. Apply point-slope form through $(1, 0)$:

$$y - 0 = \frac{3}{2}(x - 1).$$

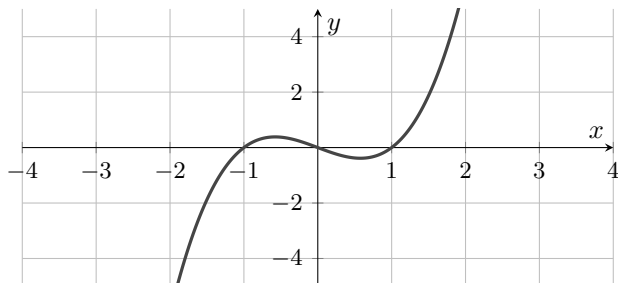
Therefore, the only tangent line at the specified point is

$$\boxed{y = \frac{3}{2}(x - 1)}.$$

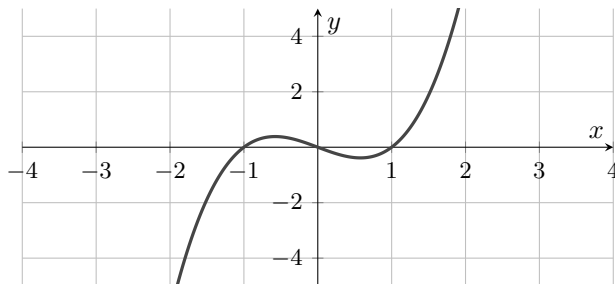
Question 10

Problem

Design a so that $x = t$, $y = t^3 - at$ meets the x -axis at $t = 1$ with slope 2. Is it possible?



Question 10 – Solution



Solution

1. To meet the x -axis at $t = 1$, the y -coordinate must be zero:

$$y(1) = 1^3 - a(1) = 1 - a = 0.$$

Therefore, the position condition requires

$$a = 1.$$

2. Now impose the slope condition. Differentiate:

$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = 3t^2 - a.$$

Since $dx/dt = 1$, the tangent slope is

$$\frac{dy}{dx} = 3t^2 - a.$$

3. At $t = 1$, requiring slope 2 gives

$$3 - a = 2 \implies a = 1.$$

Thus the position and slope conditions agree.

4. Verify the result directly. With $a = 1$,

$$y(1) = 1 - 1 = 0, \quad \left. \frac{dy}{dx} \right|_{t=1} = 3 - 1 = 2.$$

The point is $(x(1), y(1)) = (1, 0)$, and its tangent line is $y = 2(x - 1)$.

5. Hence the design is possible, and the unique value is

$$\boxed{a = 1}.$$