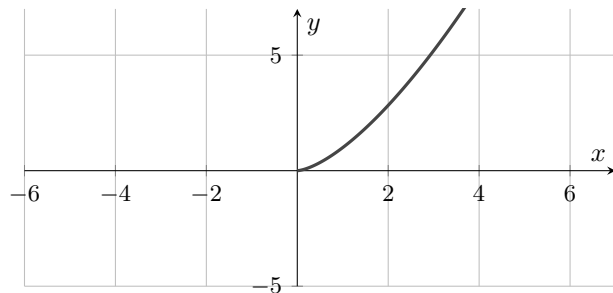


Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 1

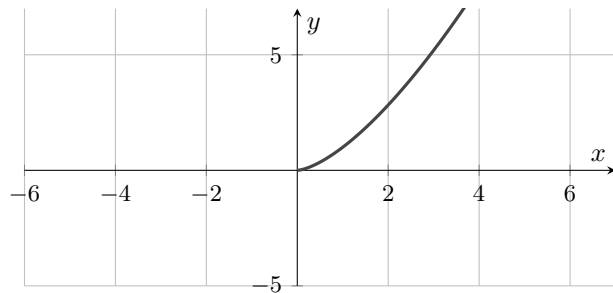
Problem

For $y = x^{3/2}$, $0 \leq x \leq 4$, compare the Cartesian and parametric arc-length setups using $x = t^2$, $y = t^3$.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 1 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

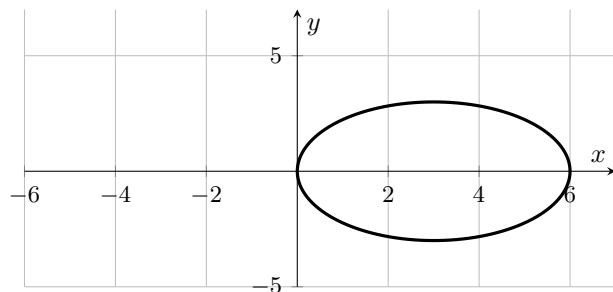
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian: $\int_0^4 \sqrt{1 + 9x/4} dx$.
4. Parametric with $0 \leq t \leq 2$: $\int_0^2 t\sqrt{4 + 9t^2} dt$.
5. Substitution $x = t^2$ makes them identical.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

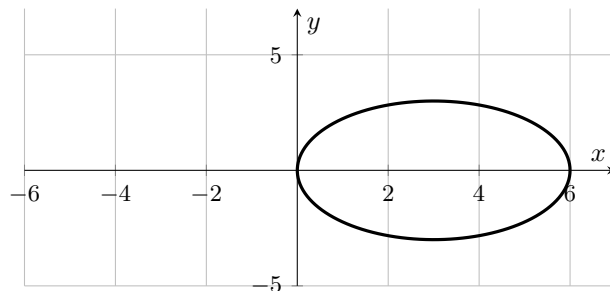
Question 2

Problem

A circle is given parametrically and polarly. Which representation makes its circumference easiest: $x = 3 + 3 \cos t$, $y = 3 \sin t$ or $r = 6 \cos \theta$? Explain.



Question 2 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

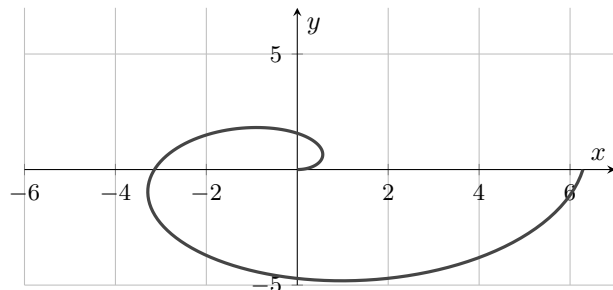
$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametric speed is 3 on $0 \leq t \leq 2\pi$. The polar arc-length factor is 6 on $-\pi/2 \leq \theta \leq \pi/2$. Each interval traces the circle once, so both forms give $L = 6\pi$.
4. The best choice is the one whose single-tracing interval is clearest.

Question 3

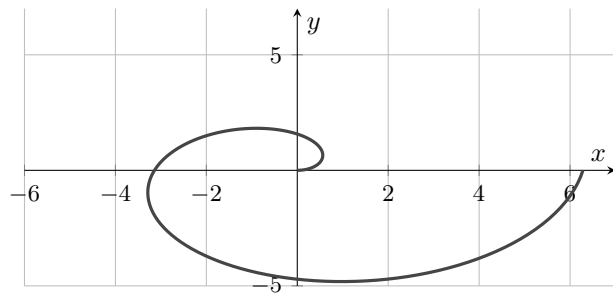
Problem

The spiral $r = \theta$ can be parametrized by $x = \theta \cos \theta, y = \theta \sin \theta$. Verify that both arc-length formulas give the same integrand.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 3 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

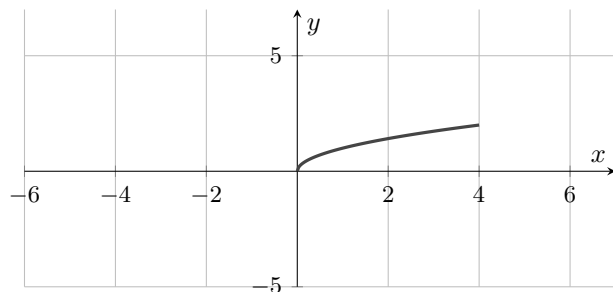
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Polar gives $\sqrt{r^2 + (r')^2} = \sqrt{\theta^2 + 1}$.
4. Squaring the Cartesian derivatives cancels cross terms and gives the same $\sqrt{\theta^2 + 1}$.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 4

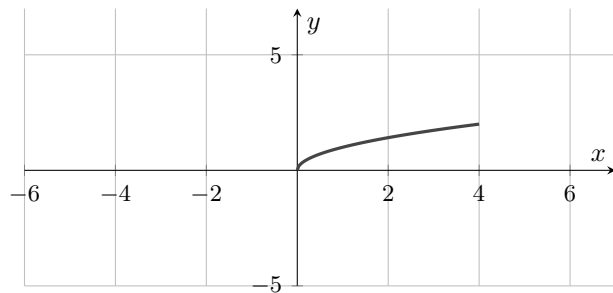
Problem

A surface is generated from $y = \sqrt{x}$, $0 \leq x \leq 4$, about the x -axis. Decide whether Cartesian or the parametrization $x = t^2, y = t$ is cleaner.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 4 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The parametrization removes the singular derivative at $x = 0$:

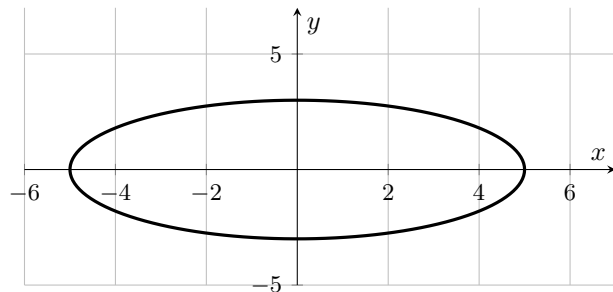
$$ds = \sqrt{4t^2 + 1} dt, \quad S = 2\pi \int_0^2 t\sqrt{4t^2 + 1} dt.$$

Therefore the parametric form is cleaner.

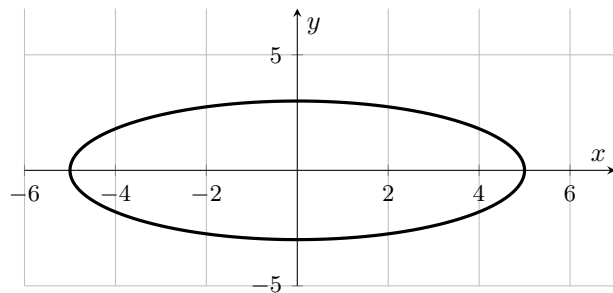
Question 5

Problem

A noncircular ellipse lacks an elementary perimeter formula. Explain why switching among Cartesian, parametric, and polar forms does not remove the essential difficulty.



Question 5 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

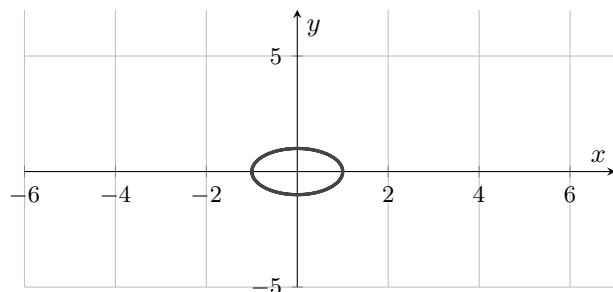
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Each representation reduces to an elliptic integral.
4. Reparametrization changes appearance, not the intrinsic arc-length integral or its non-elementary nature.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 6

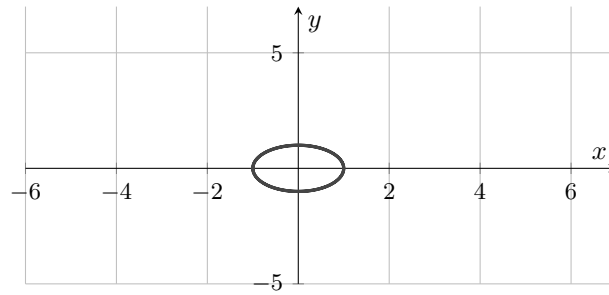
Problem

A curve is traced twice by a parametrization but once by a polar interval. Which arc length is the geometric length, and which is the traveled distance?



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 6 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

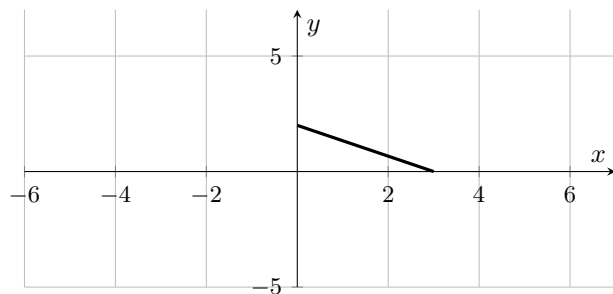
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. The once-traced polar integral gives geometric curve length.
4. The twice-traced parametric integral gives total traveled distance, twice the geometric length.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

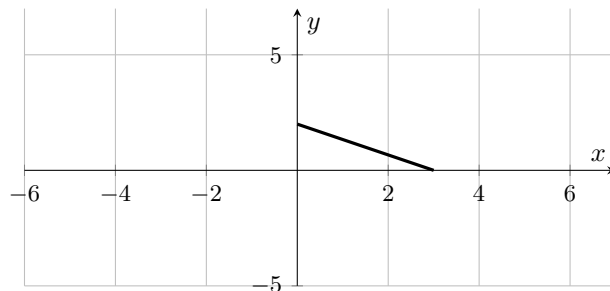
Question 7

Problem

Recover the cone lateral-area formula by parametrizing its slanted edge, then compare it with the slant-height mnemonic.



Question 7 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

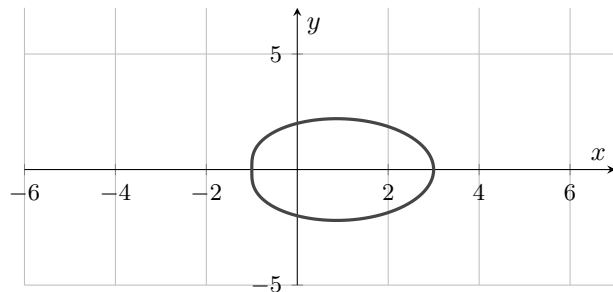
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. For radius $R > 0$ and height $h > 0$, rotate $(x, y) = (Rt/h, h - t)$, $0 \leq t \leq h$, about the y -axis.
4. Then $ds = (\sqrt{R^2 + h^2}/h)dt$ and $S = \pi R\sqrt{R^2 + h^2} = \pi R\ell$, matching the mnemonic.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 8

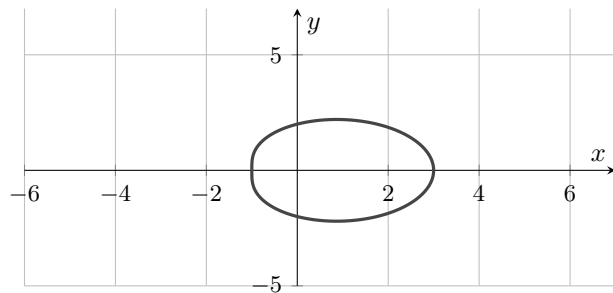
Problem

A curve has polar form $r = f(\theta)$ but is rotated about the y -axis. State when converting to Cartesian form might be safer.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 8 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

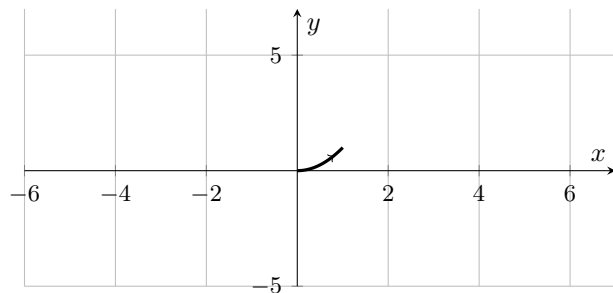
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Conversion is safer when $x = r \cos \theta$ changes sign or the polar interval retraces points, because the radius to the y -axis is $|x|$ and double-covering is easier to see geometrically.

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 9

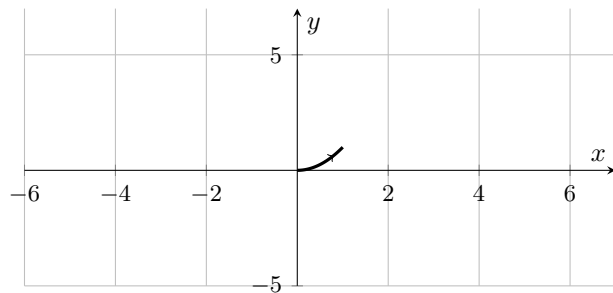
Problem

Design a consistency check for any computed surface area of revolution using dimensions and scaling.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 9 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

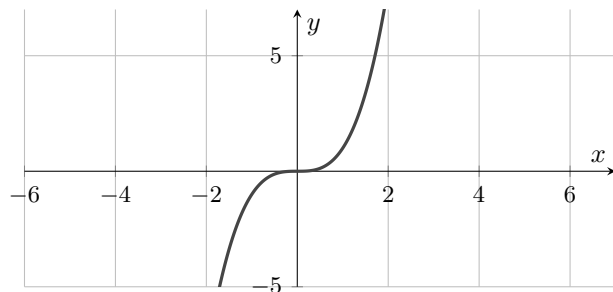
2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Area must have length-squared units.
4. If every coordinate is scaled by $k > 0$, both radius and arc length scale by k , so the answer must scale by k^2 .

Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 10

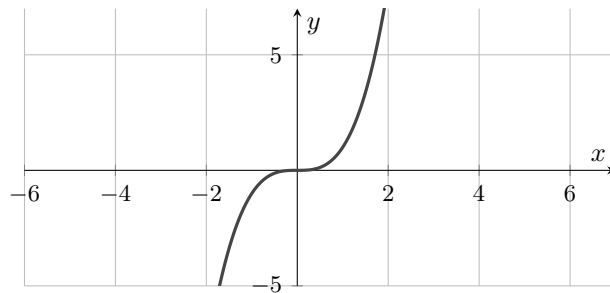
Problem

A numerical system can evaluate either a Cartesian or parametric arc-length integral. Give one reason to prefer each form.



Calculus II Section 3-11: Arc Length and Surface Area Revisited

Question 10 – Solution



Solution

1. Choose a representation and interval that trace the desired curve exactly once. The equivalent arc-length formulas are

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt,$$

$$L = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. For a surface of revolution, multiply the appropriate arc-length element by 2π times the nonnegative distance to the axis. Check the tracing interval to prevent geometric double-counting.
3. Cartesian is convenient for a single-valued smooth graph with simple derivative.
4. Parametric form is preferable for loops, vertical tangents, motion data, or when it avoids an infinite derivative.