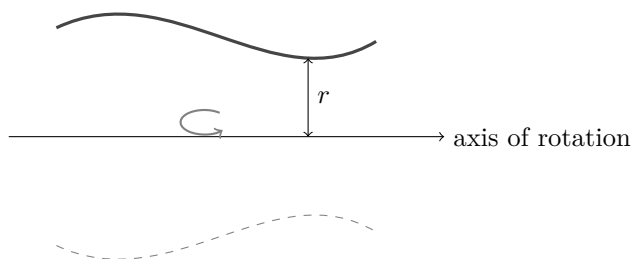


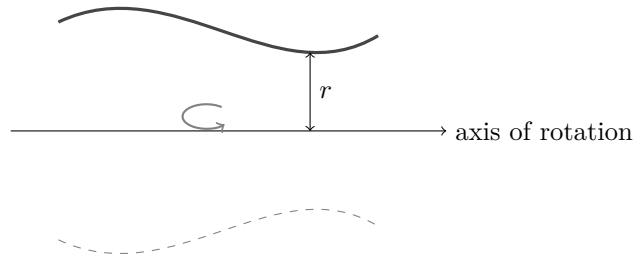
Question 1

Problem

The upper semicircle $r = 2R \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the polar axis. Find the generated surface area.



Question 1 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

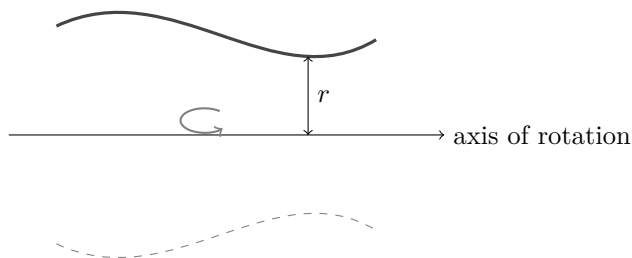
3. Here $y = 2R \sin \theta \cos \theta$ and $ds = 2R d\theta$.

4. Thus $S = 8\pi R^2 \int_0^{\pi/2} \sin \theta \cos \theta d\theta = \boxed{4\pi R^2}$.

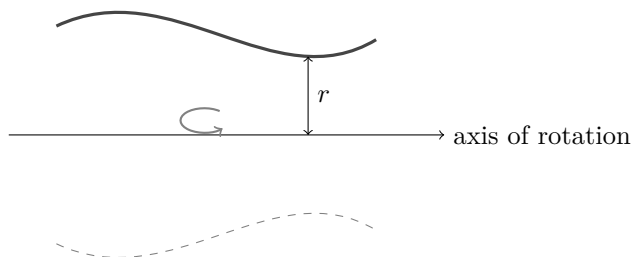
Question 2

Problem

Set up the surface area formed by rotating the cardioid $r = 1 + \cos \theta$, $0 \leq \theta \leq \pi$, about the polar axis.



Question 2 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

3. Use $y = r \sin \theta$ and $ds = 2 \cos(\theta/2) d\theta$ on this interval.

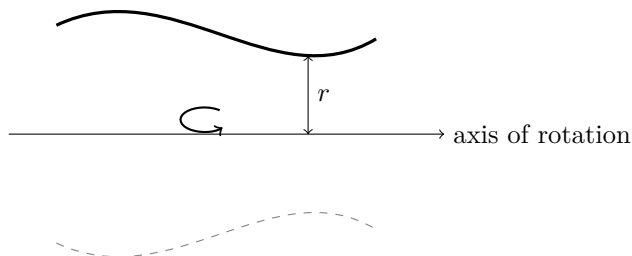
4. Therefore

$$S = 4\pi \int_0^\pi (1 + \cos \theta) \sin \theta \cos(\theta/2) d\theta.$$

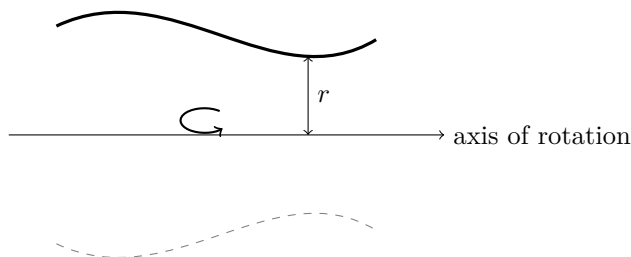
Question 3

Problem

Rotate the spiral arc $r = \theta$, $0 \leq \theta \leq \pi/2$, about the y -axis. Write the exact surface-area integral.



Question 3 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

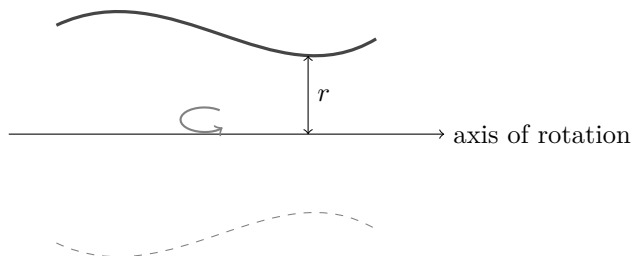
3. Distance to the y -axis is $x = \theta \cos \theta$, and $ds = \sqrt{1 + \theta^2} d\theta$.

4. Thus
$$S = 2\pi \int_0^{\pi/2} \theta \cos \theta \sqrt{1 + \theta^2} d\theta.$$

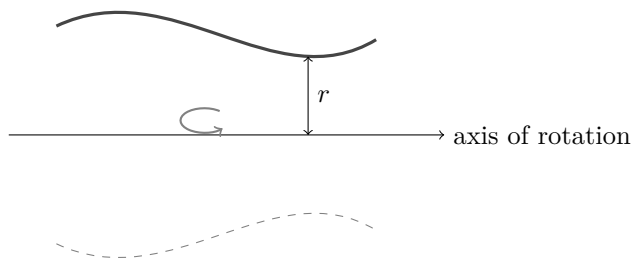
Question 4

Problem

Why must $|r \sin \theta|$ replace $r \sin \theta$ when a polar curve crosses the x -axis during rotation about that axis?



Question 4 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

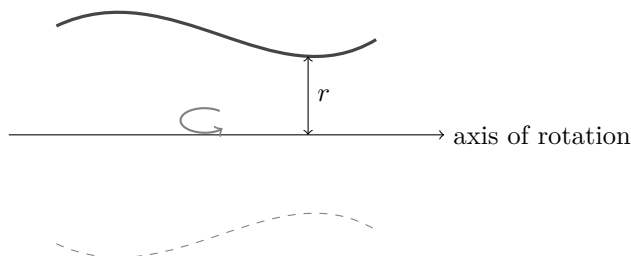
over an interval that generates the surface exactly once.

3. Radius to the axis is the nonnegative distance $|y| = |r \sin \theta|$.
4. Without the absolute value, areas on opposite sides would cancel.

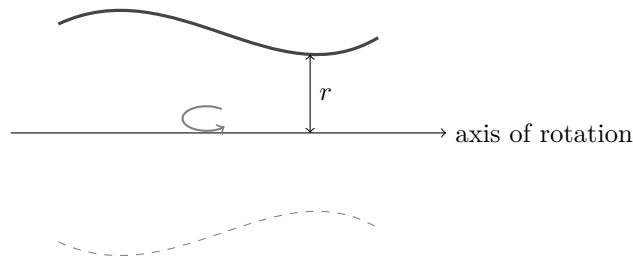
Question 5

Problem

A student proposes using only $0 \leq \theta \leq \pi/2$ when the circle $r = 4 \sin \theta$ rotates about the x -axis. Explain why that omits part of the surface and give a correct interval.



Question 5 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

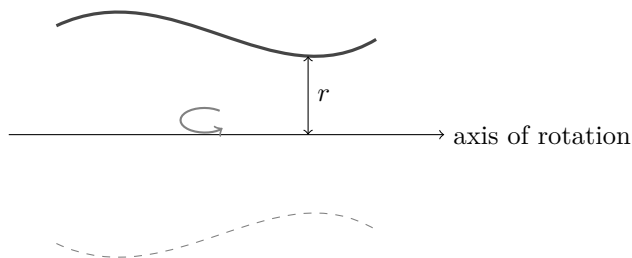
over an interval that generates the surface exactly once.

3. The interval $0 \leq \theta \leq \pi/2$ traces only the right semicircle; rotation preserves the x -coordinate, so the left half is not duplicated.
4. Use $\boxed{0 \leq \theta \leq \pi}$ to trace the full generating circle once.

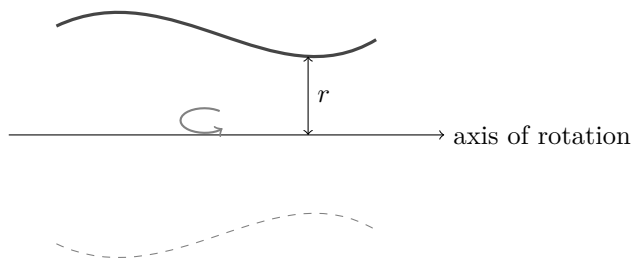
Question 6

Problem

For $r = R$, $0 \leq \theta \leq \pi$, rotate the semicircle about the x -axis and recover the sphere area.



Question 6 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

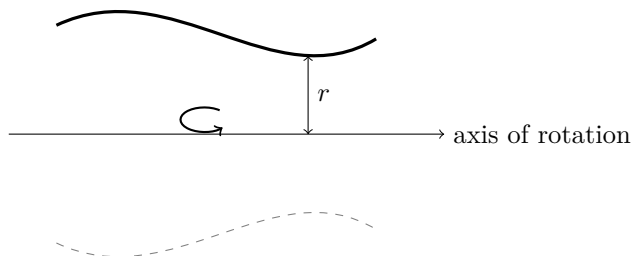
3. $y = R \sin \theta$ and $ds = R d\theta$.

4. Thus $S = 2\pi R^2 \int_0^\pi \sin \theta d\theta = \boxed{4\pi R^2}$.

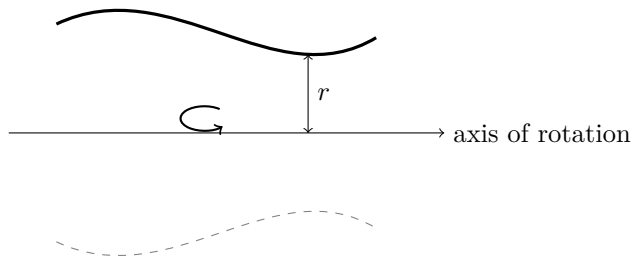
Question 7

Problem

Set up the surface area obtained by rotating one petal of $r = \cos 2\theta$, $-\pi/4 \leq \theta \leq \pi/4$, about the y -axis.



Question 7 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

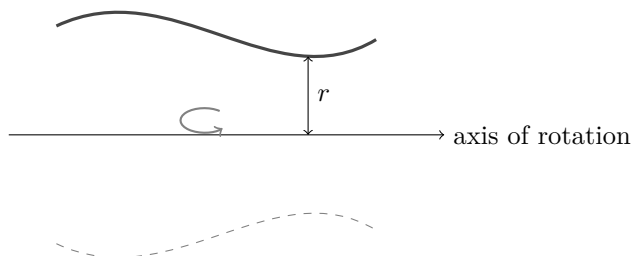
3. Use radius $|x| = |r \cos \theta|$ and $ds = \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta$.
4. On this interval $r \cos \theta \geq 0$, so the requested setup is

$$S = 2\pi \int_{-\pi/4}^{\pi/4} \cos 2\theta \cos \theta \sqrt{\cos^2 2\theta + 4 \sin^2 2\theta} d\theta.$$

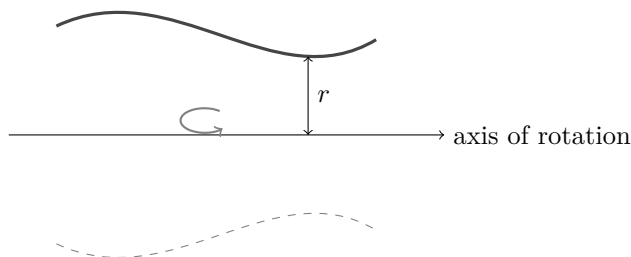
Question 8

Problem

A curve is entirely in the first quadrant. Compare its areas of revolution about the x - and y -axes using only weighted average coordinates.



Question 8 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

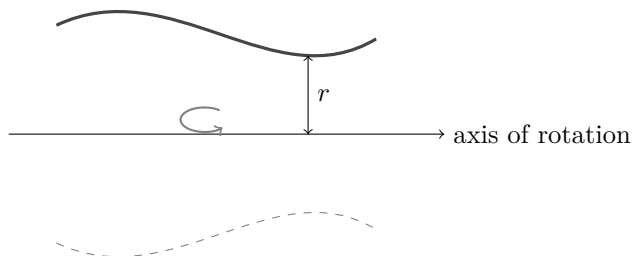
over an interval that generates the surface exactly once.

3. $S_x = 2\pi L \bar{y}_s$ and $S_y = 2\pi L \bar{x}_s$, where the bars are arc-length-weighted means.
4. Therefore $S_x/S_y = \bar{y}_s/\bar{x}_s$.

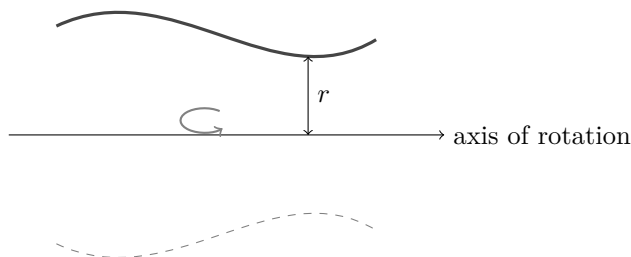
Question 9

Problem

Find the area generated when $r = 2 \cos \theta$, $0 \leq \theta \leq \pi/2$, rotates about the line $y = -1$.



Question 9 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

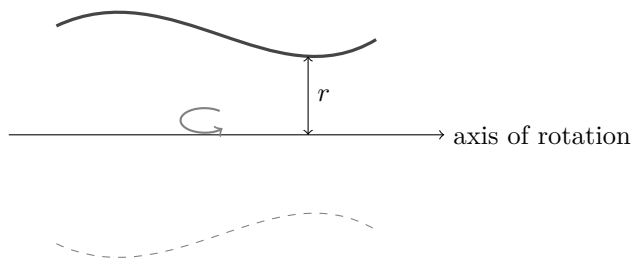
3. Here $ds = 2d\theta$ and radius is $y + 1 = 2 \sin \theta \cos \theta + 1$.

4. Hence $S = 4\pi \int_0^{\pi/2} (1 + 2 \sin \theta \cos \theta) d\theta = \boxed{2\pi^2 + 4\pi}$.

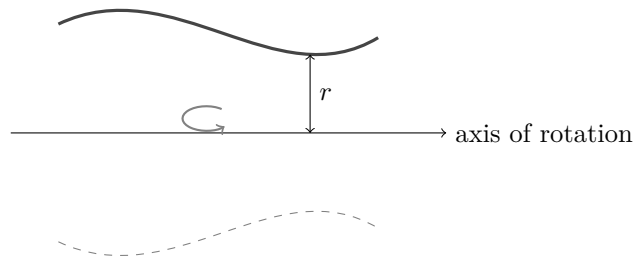
Question 10

Problem

A student integrates a closed polar curve over $0 \leq \theta \leq 2\pi$ and gets twice the expected surface area. Name two likely causes.



Question 10 – Solution



Solution

1. Compute the polar arc-length element

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta.$$

2. Express the radius of rotation as a nonnegative distance: use $|r \sin \theta|$ for the x -axis and $|r \cos \theta|$ for the y -axis. Then apply

$$S = 2\pi \int_a^b (\text{radius to the axis}) ds,$$

over an interval that generates the surface exactly once.

3. The curve may itself be traced twice on that interval, or two different portions of the generating curve may sweep the same surface after rotation.
4. Both are geometric double-counting.