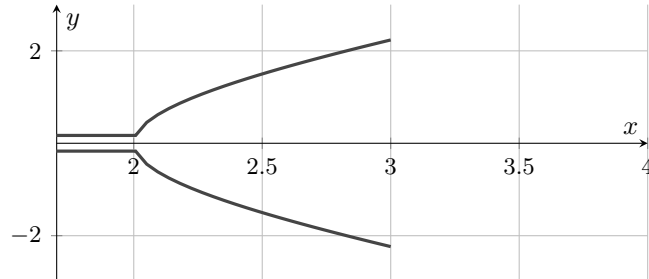


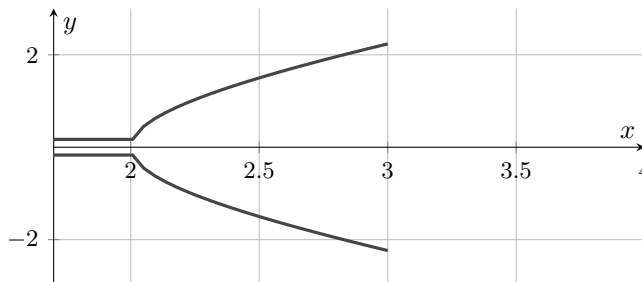
Question 1

Problem

A mechanism has $x = t + t^{-1}$ and $y = t - t^{-1}$, $t > 0$. Identify its Cartesian curve and explain which part the mechanism can reach.



Question 1 – Solution



Solution

1. Square both parametric equations:

$$x^2 = \left(t + \frac{1}{t}\right)^2 = t^2 + 2 + \frac{1}{t^2}, \quad y^2 = \left(t - \frac{1}{t}\right)^2 = t^2 - 2 + \frac{1}{t^2}.$$

2. Subtract the second equation from the first:

$$x^2 - y^2 = 4.$$

Thus the Cartesian curve is the hyperbola

$$\boxed{\frac{x^2}{4} - \frac{y^2}{4} = 1}.$$

3. Because $t > 0$, the arithmetic–geometric mean inequality gives

$$x = t + \frac{1}{t} \geq 2.$$

Therefore, the parametrization reaches only the right-hand branch of the hyperbola.

4. To determine the direction of travel, note that

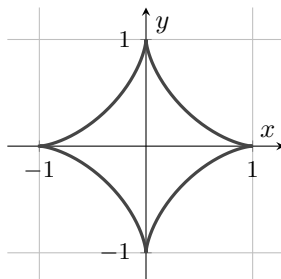
$$\frac{dx}{dt} = 1 - \frac{1}{t^2}, \quad \frac{dy}{dt} = 1 + \frac{1}{t^2} > 0.$$

As t increases from 0 to 1, the point moves from the lower right toward $(2, 0)$. As t increases beyond 1, it moves from $(2, 0)$ toward the upper right.

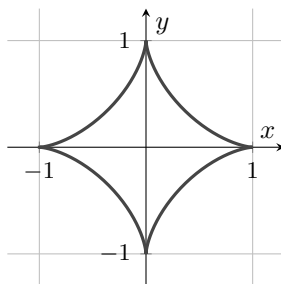
5. Hence the mechanism traces the entire right branch exactly once, moving upward as t increases.

Question 2**Problem**

The curve $x = \cos^3 t$, $y = \sin^3 t$, $0 \leq t \leq 2\pi$, looks almost circular. Eliminate t and identify the four points where its shape is least circle-like.



Question 2 – Solution



Solution

1. Begin with

$$x = \cos^3 t, \quad y = \sin^3 t.$$

Taking each coordinate to the power $2/3$ gives

$$x^{2/3} = \cos^2 t, \quad y^{2/3} = \sin^2 t.$$

Here $x^{2/3}$ means $(\sqrt[3]{x})^2$, so the calculation is valid even when a coordinate is negative.

2. Add the two equations and use $\cos^2 t + \sin^2 t = 1$:

$$\boxed{x^{2/3} + y^{2/3} = 1}.$$

This curve is an *astroid*.

3. The curve is least circle-like where it has cusps. These occur when one trigonometric coordinate is 0 and the other is ± 1 .
4. Evaluating the parametrization at the corresponding parameter values gives

t	(x, y)
0	(1, 0)
$\pi/2$	(0, 1)
π	(-1, 0)
$3\pi/2$	(0, -1)

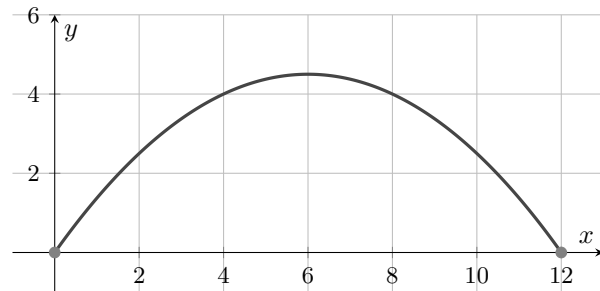
5. Therefore, the four sharp points are

$$\boxed{(1, 0), (0, 1), (-1, 0), (0, -1)}.$$

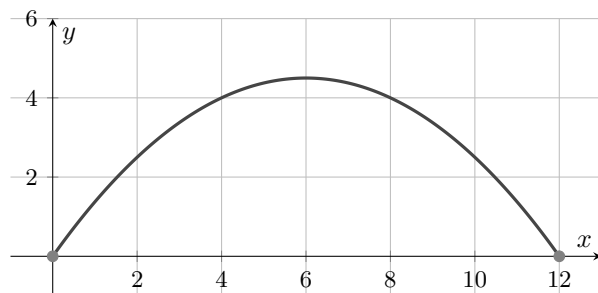
Question 3

Problem

A fountain droplet follows $x = 6t$ and $y = 9t - \frac{9}{2}t^2$. Eliminate time, find the landing point, and state why the parameter is more informative than the Cartesian equation.



Question 3 – Solution



Solution

1. Solve the horizontal equation for the parameter:

$$x = 6t \implies t = \frac{x}{6}.$$

2. Substitute $t = x/6$ into the vertical equation:

$$\begin{aligned} y &= 9\left(\frac{x}{6}\right) - \frac{9}{2}\left(\frac{x}{6}\right)^2 \\ &= \frac{3}{2}x - \frac{1}{8}x^2. \end{aligned}$$

Thus the Cartesian path is the downward-opening parabola

$$\boxed{y = \frac{3}{2}x - \frac{1}{8}x^2}.$$

3. The droplet lands when it returns to ground level, so set $y = 0$:

$$0 = \frac{3}{2}x - \frac{1}{8}x^2 = \frac{x}{8}(12 - x).$$

The solutions are $x = 0$ and $x = 12$.

4. The value $x = 0$ is the launch point. The nonzero intercept is therefore the landing point

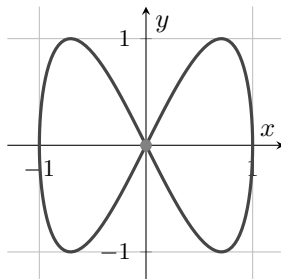
$$\boxed{(12, 0)}.$$

It occurs at $t = 12/6 = 2$.

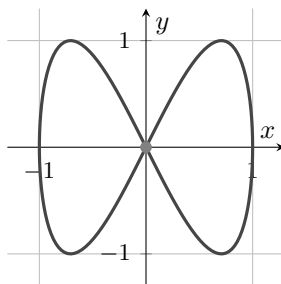
5. The Cartesian equation describes only the geometric path. The parameter t also records time, so it identifies the droplet's position and direction of motion at each instant.

Question 4**Problem**

For $x = \cos t$, $y = \sin(2t)$, $0 \leq t \leq 2\pi$, locate the self-intersection and determine how many times it is visited.



Question 4 – Solution



Solution

1. A self-intersection occurs when two different parameter values produce the same point. The graph suggests checking the origin.
2. For the point to lie at the origin, first require $x = 0$:

$$\cos t = 0.$$

On $0 \leq t \leq 2\pi$, this gives

$$t = \frac{\pi}{2} \quad \text{or} \quad t = \frac{3\pi}{2}.$$

3. Check the y -coordinate at both values:

$$y\left(\frac{\pi}{2}\right) = \sin \pi = 0, \quad y\left(\frac{3\pi}{2}\right) = \sin 3\pi = 0.$$

Hence both parameter values produce $(0, 0)$.

4. The two visits correspond to different directions. Since

$$\frac{dy}{dx} = \frac{2 \cos(2t)}{-\sin t},$$

the slopes are 2 at $t = \pi/2$ and -2 at $t = 3\pi/2$.

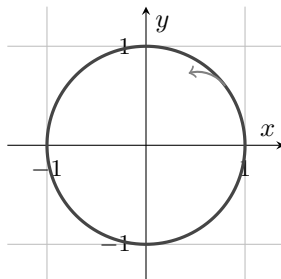
5. Therefore, the curve crosses itself at

$$\boxed{(0, 0)},$$

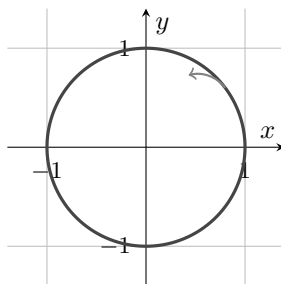
and this point is visited exactly twice on the stated interval.

Question 5**Problem**

Two students parametrize the unit circle by $(\cos t, \sin t)$ and $(\cos 2t, -\sin 2t)$ on $0 \leq t \leq 2\pi$. Compare orientation and number of traversals.



Question 5 – Solution

**Solution**

1. For the first parametrization,

$$(x, y) = (\cos t, \sin t),$$

the polar angle is t . As t increases from 0 to 2π , the angle increases through one full revolution.

2. It begins at $(1, 0)$, then moves through $(0, 1)$, $(-1, 0)$, and $(0, -1)$ before returning to $(1, 0)$. Therefore, it traces the circle counterclockwise exactly once.

3. Rewrite the second parametrization as

$$(x, y) = (\cos(2t), -\sin(2t)) = (\cos(-2t), \sin(-2t)).$$

Its polar angle is therefore $-2t$.

4. As t increases from 0 to 2π , the angle decreases from 0 to -4π . A decreasing angle produces clockwise motion, and a total angular change of 4π represents two complete revolutions.

5. Thus

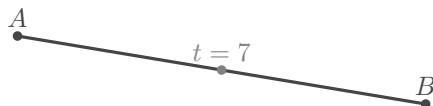
$(\cos t, \sin t)$ travels counterclockwise once,

whereas

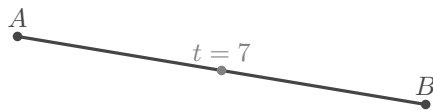
$(\cos 2t, -\sin 2t)$ travels clockwise twice.

Question 6**Problem**

Find a parametrization of the line segment from $A = (-2, 5)$ to $B = (4, -1)$ that reaches its midpoint at $t = 7$, with $6 \leq t \leq 8$.



Question 6 – Solution



Solution

1. A line segment from A to B can be parametrized by

$$\mathbf{r} = A + u(B - A), \quad 0 \leq u \leq 1.$$

2. The parameter interval must be changed from $6 \leq t \leq 8$ to $0 \leq u \leq 1$. The linear change of variables is

$$u = \frac{t - 6}{8 - 6} = \frac{t - 6}{2}.$$

3. Compute the displacement vector:

$$B - A = (4, -1) - (-2, 5) = (6, -6).$$

Therefore,

$$(x, y) = (-2, 5) + \frac{t - 6}{2}(6, -6).$$

4. Simplify each coordinate:

$$x = -2 + 3(t - 6) = 3t - 20,$$

$$y = 5 - 3(t - 6) = 23 - 3t.$$

Hence a suitable parametrization is

$$\boxed{x = 3t - 20, \quad y = 23 - 3t, \quad 6 \leq t \leq 8}.$$

5. At $t = 6$ it gives $A = (-2, 5)$, and at $t = 8$ it gives $B = (4, -1)$. At $t = 7$ it gives

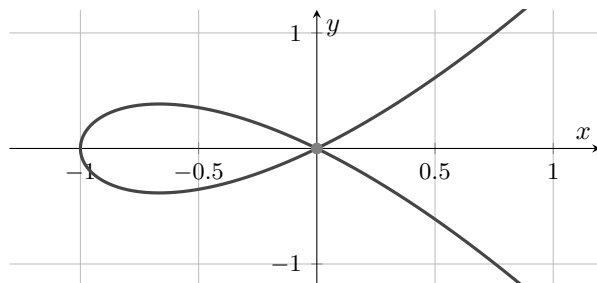
$$(x, y) = (1, 2) = \frac{A + B}{2},$$

so the midpoint condition is satisfied.

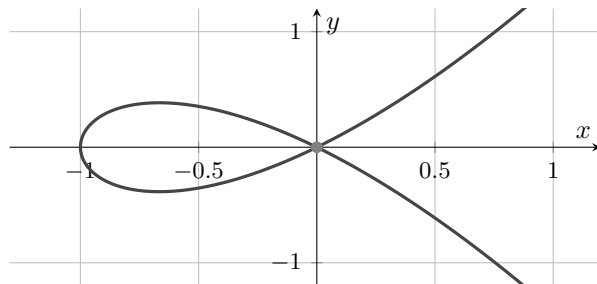
Question 7

Problem

The curve $x = t^2 - 1$, $y = t^3 - t$ crosses itself. Find the two parameter values at the crossing and describe the two tangent directions there.



Question 7 – Solution



Solution

1. Factor the coordinate equations:

$$x = t^2 - 1 = (t - 1)(t + 1), \quad y = t^3 - t = t(t - 1)(t + 1).$$

2. Both coordinates equal zero when $t = 1$ and when $t = -1$:

$$\mathbf{r}(1) = (0, 0), \quad \mathbf{r}(-1) = (0, 0).$$

Because two distinct parameter values produce the same point, the self-intersection is $(0, 0)$.

3. Differentiate both coordinates:

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 - 1.$$

Since $dx/dt \neq 0$ at $t = \pm 1$, the tangent slope is

$$\frac{dy}{dx} = \frac{3t^2 - 1}{2t}.$$

4. At $t = 1$,

$$\frac{dy}{dx} = \frac{3(1)^2 - 1}{2(1)} = 1,$$

so one tangent line is $y = x$.

5. At $t = -1$,

$$\frac{dy}{dx} = \frac{3(-1)^2 - 1}{2(-1)} = -1,$$

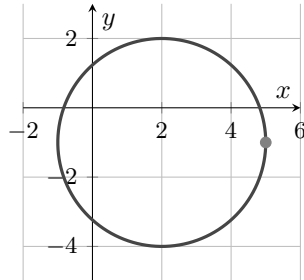
so the other tangent line is $y = -x$.

6. Therefore, the curve crosses itself at the origin with tangent directions of slopes

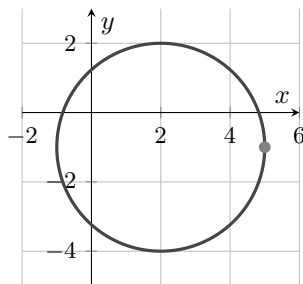
$$\boxed{1 \text{ and } -1}.$$

Question 8**Problem**

A moving point satisfies $x = 2 + 3 \cos t$, $y = -1 + 3 \sin t$. Without eliminating t , determine its center, radius, starting point, and initial direction.



Question 8 – Solution

**Solution**

1. Compare the equations

$$x = 2 + 3 \cos t, \quad y = -1 + 3 \sin t$$

with the standard circle parametrization

$$x = h + R \cos t, \quad y = k + R \sin t.$$

2. It follows immediately that

$$(h, k) = (2, -1) \quad \text{and} \quad R = 3.$$

Thus the curve is a circle centered at $(2, -1)$ with radius 3 .

3. Evaluate the coordinates at $t = 0$:

$$x(0) = 2 + 3 \cos 0 = 5, \quad y(0) = -1 + 3 \sin 0 = -1.$$

Therefore, the starting point is $(5, -1)$.

4. Differentiate to find the velocity:

$$\mathbf{r}'(t) = (-3 \sin t, 3 \cos t).$$

At $t = 0$,

$$\mathbf{r}'(0) = (0, 3),$$

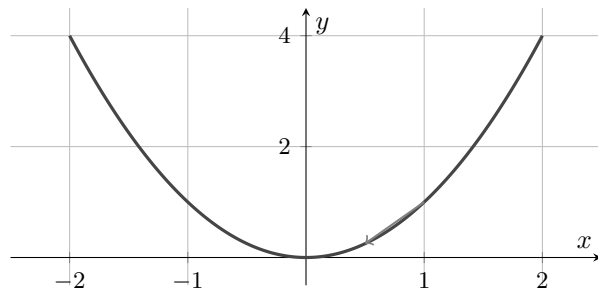
which points straight upward from the rightmost point of the circle.

5. Upward motion from the rightmost point is counterclockwise, so the initial direction is **counterclockwise**.

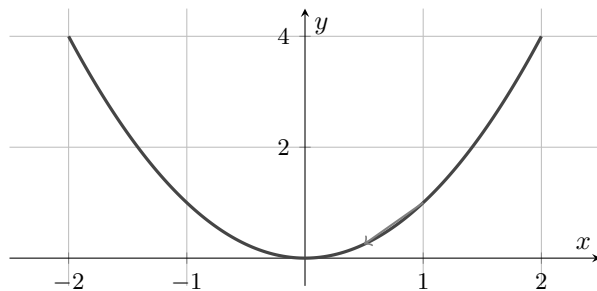
Question 9

Problem

Can the parabola $y = x^2$ be traced from right to left while the parameter increases? Give a parametrization on $-2 \leq t \leq 2$ that does so.



Question 9 – Solution



Solution

1. To trace the parabola from right to left, choose the x -coordinate so that it decreases as t increases. A simple choice is

$$x = -t.$$

2. The curve must satisfy $y = x^2$. Substituting $x = -t$ gives

$$y = (-t)^2 = t^2.$$

Hence one suitable parametrization is

$$\boxed{x = -t, \quad y = t^2, \quad -2 \leq t \leq 2}.$$

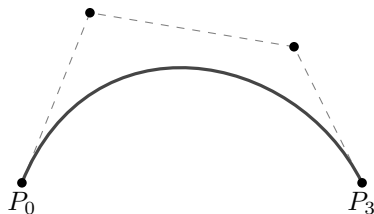
3. Verify that the curve is correct by eliminating the parameter:

$$t = -x \implies y = t^2 = (-x)^2 = x^2.$$

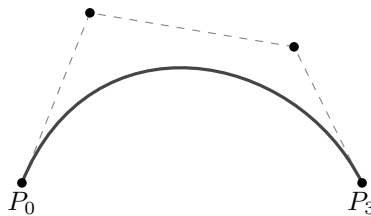
4. Verify the direction. At $t = -2$, the point is $(2, 4)$; at $t = 0$, it is $(0, 0)$; and at $t = 2$, it is $(-2, 4)$.
5. Also, $dx/dt = -1 < 0$, so x decreases continuously. Therefore, the parabola is traced from right to left as required.

Question 10**Problem**

A cubic Bézier path is $\mathbf{r}(t) = (1-t)^3P_0 + 3(1-t)^2tP_1 + 3(1-t)t^2P_2 + t^3P_3$. Explain, without expanding, why it begins at P_0 and ends at P_3 .



Question 10 – Solution



Solution

1. The cubic Bézier path is

$$\mathbf{r}(t) = (1-t)^3 P_0 + 3(1-t)^2 t P_1 + 3(1-t) t^2 P_2 + t^3 P_3.$$

2. Evaluate the path at $t = 0$:

$$\begin{aligned}\mathbf{r}(0) &= (1-0)^3 P_0 + 3(1-0)^2 (0) P_1 \\ &\quad + 3(1-0)(0)^2 P_2 + (0)^3 P_3 \\ &= P_0.\end{aligned}$$

Every term containing a factor of t vanishes, leaving only P_0 .

3. Evaluate the path at $t = 1$:

$$\begin{aligned}\mathbf{r}(1) &= (1-1)^3 P_0 + 3(1-1)^2 (1) P_1 \\ &\quad + 3(1-1)(1)^2 P_2 + (1)^3 P_3 \\ &= P_3.\end{aligned}$$

Every term containing a factor of $1-t$ vanishes, leaving only P_3 .

4. Therefore,

$$\boxed{\mathbf{r}(0) = P_0 \quad \text{and} \quad \mathbf{r}(1) = P_3}.$$

The path begins at the first control point and ends at the last control point.