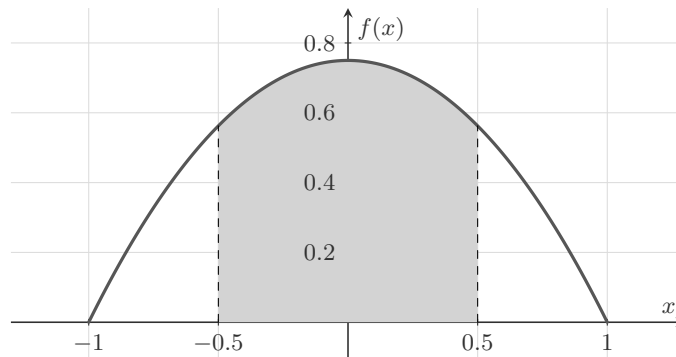
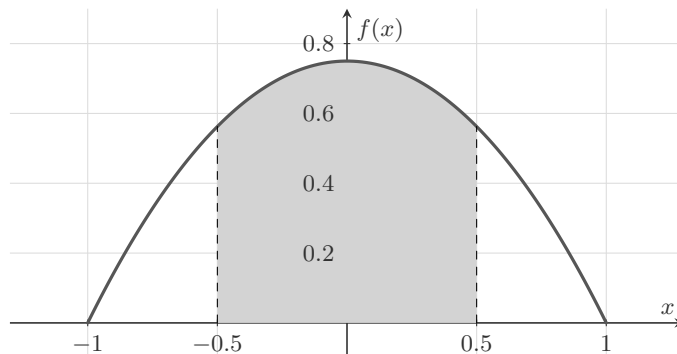


Question 1

A sensor error has density $f(x) = c(1 - x^2)$, $-1 \leq x \leq 1$. Find c and $P(|X| < 1/2)$.



Question 1 – Solution



Step 1: Use the normalization condition. A probability density must have total area 1:

$$1 = c \int_{-1}^1 (1 - x^2) dx.$$

Step 2: Evaluate the normalization integral.

$$\begin{aligned} \int_{-1}^1 (1 - x^2) dx &= \left[x - \frac{x^3}{3} \right]_{-1}^1 \\ &= \frac{2}{3} - \left(-\frac{2}{3} \right) = \frac{4}{3}. \end{aligned}$$

Thus

$$1 = c \left(\frac{4}{3} \right), \quad \boxed{c = \frac{3}{4}}.$$

Step 3: Translate the event.

$$|X| < \frac{1}{2} \iff -\frac{1}{2} < X < \frac{1}{2}.$$

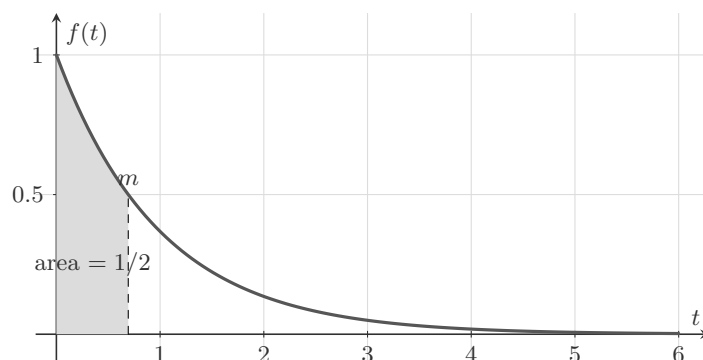
Step 4: Integrate the density over this interval.

$$\begin{aligned} P\left(|X| < \frac{1}{2}\right) &= \frac{3}{4} \int_{-1/2}^{1/2} (1 - x^2) dx \\ &= \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{-1/2}^{1/2} \\ &= \frac{3}{4} \left(\frac{11}{12} \right) = \frac{11}{16}. \end{aligned}$$

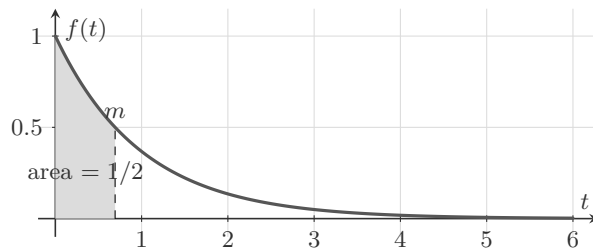
$$\boxed{P\left(|X| < \frac{1}{2}\right) = \frac{11}{16}}$$

Question 2

A waiting time has $f(t) = \lambda e^{-\lambda t}$, $t \geq 0$. Find the median in terms of λ .



Question 2 – Solution



Step 1: Use the definition of a median. The median m satisfies

$$P(T \leq m) = \frac{1}{2}.$$

Because $T \geq 0$,

$$\int_0^m \lambda e^{-\lambda t} dt = \frac{1}{2}.$$

Step 2: Evaluate the integral.

$$\begin{aligned} \int_0^m \lambda e^{-\lambda t} dt &= [-e^{-\lambda t}]_0^m \\ &= 1 - e^{-\lambda m}. \end{aligned}$$

Therefore,

$$1 - e^{-\lambda m} = \frac{1}{2}.$$

Step 3: Solve for m .

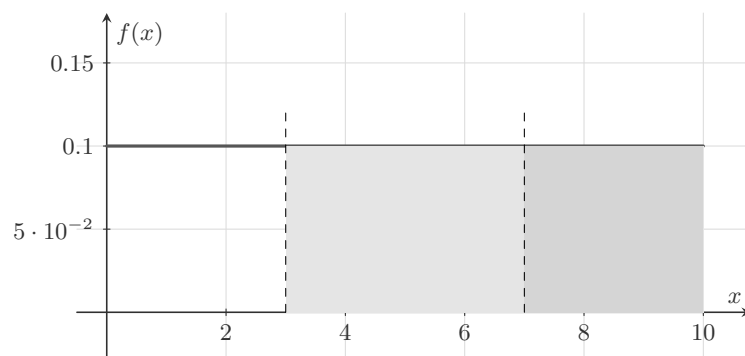
$$\begin{aligned} e^{-\lambda m} &= \frac{1}{2}, \\ -\lambda m &= \ln\left(\frac{1}{2}\right) = -\ln 2, \\ m &= \frac{\ln 2}{\lambda}. \end{aligned}$$

$$\boxed{m = \frac{\ln 2}{\lambda}}$$

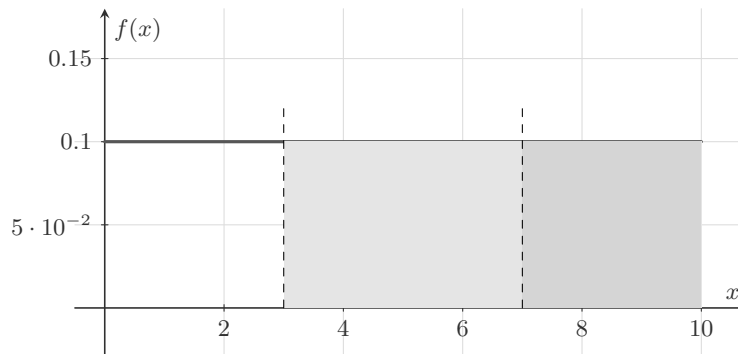
Since $\lambda > 0$, the median is positive as expected.

Question 3

A random point is chosen uniformly on $[0, 10]$. Given $X > 3$, find $P(X > 7 \mid X > 3)$.



Question 3 – Solution



Step 1: Identify the two events. Let

$$A = \{X > 7\}, \quad B = \{X > 3\}.$$

Because $A \subset B$, we have $A \cap B = A$.

Step 2: Apply conditional probability.

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}.$$

Step 3: Use uniform interval lengths. On $[0, 10]$, probability equals interval length divided by 10:

$$P(X > 7) = \frac{10 - 7}{10} = \frac{3}{10}, \quad P(X > 3) = \frac{10 - 3}{10} = \frac{7}{10}.$$

Step 4: Form the ratio.

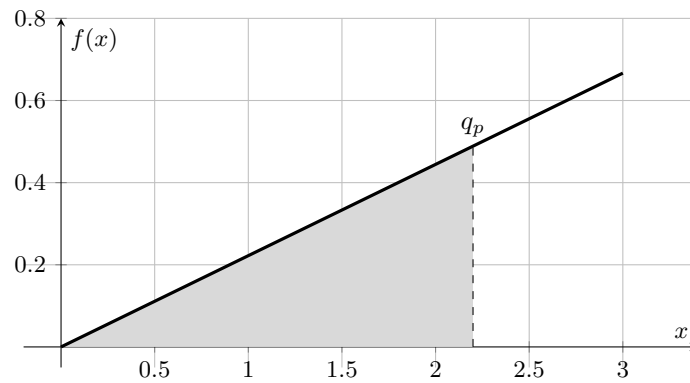
$$P(X > 7 | X > 3) = \frac{\frac{3}{10}}{\frac{7}{10}} = \frac{3}{7}.$$

$$\boxed{P(X > 7 | X > 3) = \frac{3}{7}}$$

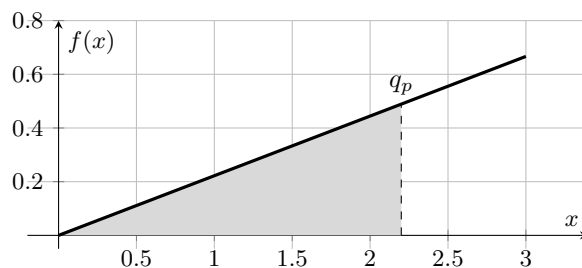
Equivalently, after conditioning on $X > 3$, the available interval is $(3, 10]$, and the favorable portion $(7, 10]$ has length 3 out of 7.

Question 4

A density is $f(x) = kx$ on $[0, a]$, where $a > 0$, and is zero elsewhere. Determine k , the CDF, and the p -quantile.



Question 4 – Solution



Step 1: Normalize the density.

$$1 = \int_0^a kx \, dx = k \left[\frac{x^2}{2} \right]_0^a = \frac{ka^2}{2}.$$

Hence

$$\boxed{k = \frac{2}{a^2}}.$$

Step 2: Build the CDF on the support. For $0 \leq x \leq a$,

$$\begin{aligned} F(x) &= \int_0^x \frac{2t}{a^2} \, dt \\ &= \frac{2}{a^2} \left[\frac{t^2}{2} \right]_0^x = \frac{x^2}{a^2}. \end{aligned}$$

Thus the complete CDF is

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{x^2}{a^2}, & 0 \leq x \leq a, \\ 1, & x > a. \end{cases}$$

Step 3: Solve the quantile equation. The p -quantile q_p satisfies $F(q_p) = p$:

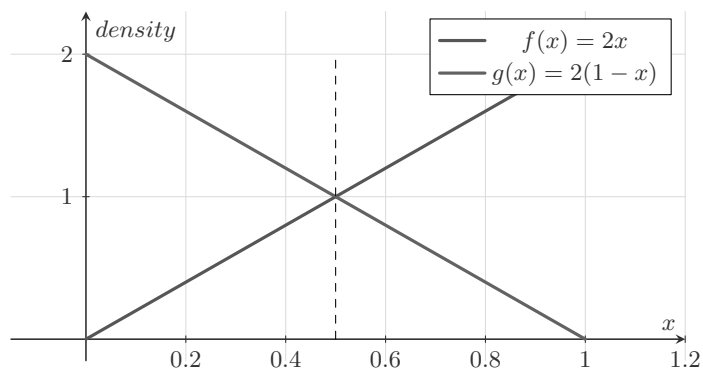
$$\frac{q_p^2}{a^2} = p.$$

Since $q_p \geq 0$,

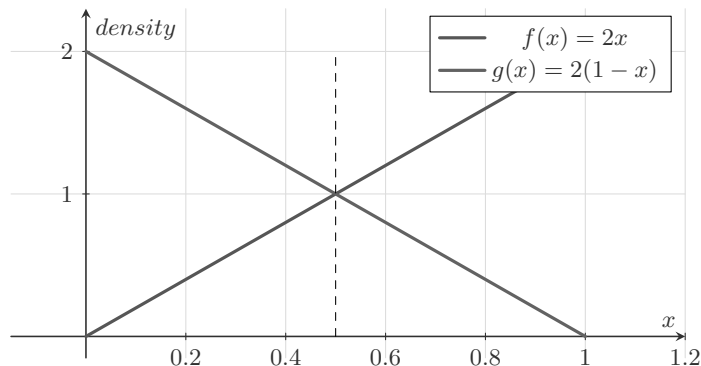
$$\boxed{q_p = a\sqrt{p}}, \quad 0 \leq p \leq 1.$$

Question 5

Two densities on $[0, 1]$ are $f(x) = 2x$ and $g(x) = 2(1 - x)$. Compare their means without integration by identifying a reflection.



Question 5 – Solution



Step 1: Identify the reflection.

$$f(1-x) = 2(1-x) = g(x).$$

Therefore, the graph of g is the reflection of the graph of f across $x = 1/2$.

Step 2: Find the mean of f geometrically. The area under $f(x) = 2x$ is a right triangle with vertices

$$(0,0), \quad (1,0), \quad (1,2).$$

The x -coordinate of a triangle's centroid is the average of its vertex x -coordinates:

$$\mu_f = \frac{0 + 1 + 1}{3} = \frac{2}{3}.$$

Step 3: Reflect the mean. Reflection across $x = 1/2$ sends a location x to $1-x$. Hence

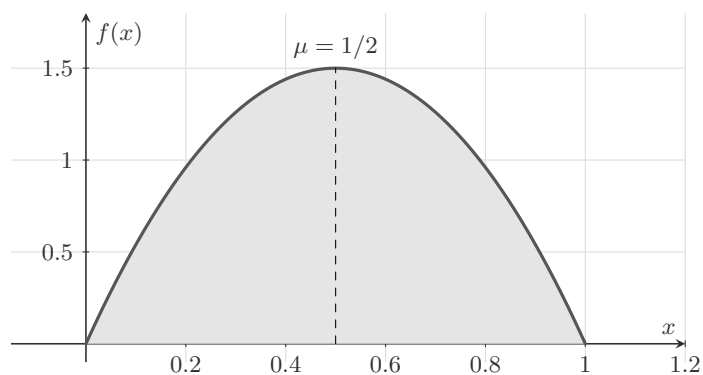
$$\mu_g = 1 - \mu_f = 1 - \frac{2}{3} = \frac{1}{3}.$$

$$\boxed{\mu_f = \frac{2}{3}, \quad \mu_g = \frac{1}{3}}$$

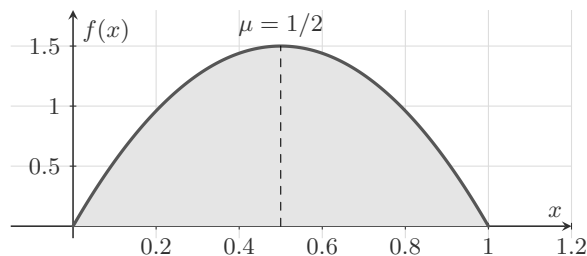
As a check, the two reflected means add to 1.

Question 6

For $f(x) = 6x(1 - x)$, $0 \leq x \leq 1$, find $\text{Var}(X)$. Use symmetry for the mean.



Question 6 – Solution



Step 1: Use symmetry to find the mean. The density satisfies

$$f(1-x) = 6(1-x)x = f(x),$$

so it is symmetric about $x = 1/2$. Therefore,

$$\mu = E[X] = \frac{1}{2}.$$

Step 2: Compute the second moment.

$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx \\ &= \int_0^1 x^2 (6x - 6x^2) dx \\ &= \int_0^1 (6x^3 - 6x^4) dx \\ &= \left[\frac{3}{2}x^4 - \frac{6}{5}x^5 \right]_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}. \end{aligned}$$

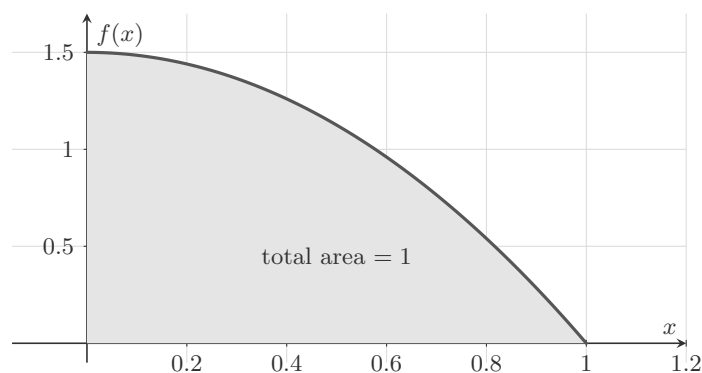
Step 3: Apply the variance formula.

$$\begin{aligned} \text{Var}(X) &= E[X^2] - (E[X])^2 \\ &= \frac{3}{10} - \left(\frac{1}{2}\right)^2 \\ &= \frac{3}{10} - \frac{1}{4} = \frac{1}{20}. \end{aligned}$$

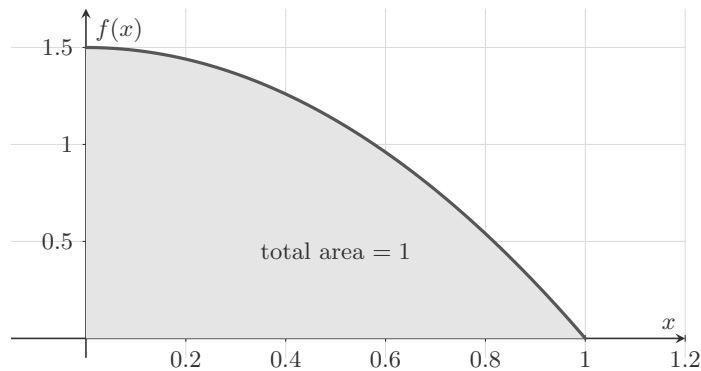
$\text{Var}(X) = \frac{1}{20}$

Question 7

A proposed density is $f(x) = \frac{3}{2}(1 - x^2)$ on $[0, 1]$. Is it valid? If not, repair only the constant.



Question 7 – Solution



Step 1: Check nonnegativity. For $0 \leq x \leq 1$,

$$1 - x^2 \geq 0,$$

so

$$f(x) = \frac{3}{2}(1 - x^2) \geq 0.$$

Step 2: Check the total area.

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{3}{2} \int_0^1 (1 - x^2) dx \\ &= \frac{3}{2} \left[x - \frac{x^3}{3} \right]_0^1 \\ &= \frac{3}{2} \left(1 - \frac{1}{3} \right) = \frac{3}{2} \left(\frac{2}{3} \right) = 1. \end{aligned}$$

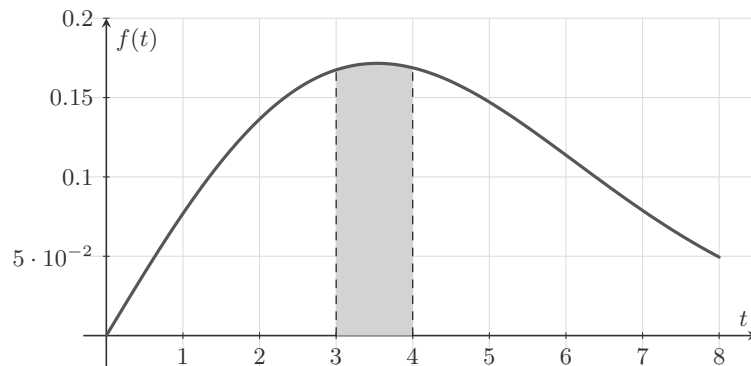
Step 3: State the conclusion. The function is nonnegative on its support and integrates to 1. Therefore it already satisfies both requirements for a probability density.

The proposed density is valid; no repair is needed.

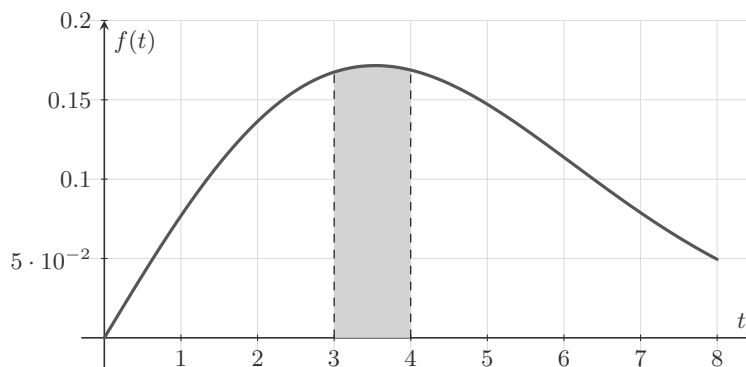
If the constant had been unknown, normalization would give $c(2/3) = 1$, hence $c = 3/2$.

Question 8

A lifetime satisfies $P(X > t) = e^{-(t/5)^2}$, $t \geq 0$. Find its density and $P(3 < X < 4)$.



Question 8 – Solution



Step 1: Convert the survival function to the CDF. For $t \geq 0$,

$$P(X > t) = e^{-(t/5)^2} = e^{-t^2/25}.$$

Therefore,

$$F(t) = P(X \leq t) = 1 - P(X > t) = 1 - e^{-t^2/25}.$$

Step 2: Differentiate to obtain the density.

$$\begin{aligned} f(t) &= F'(t) \\ &= -\frac{d}{dt} \left(e^{-t^2/25} \right) \\ &= \frac{2t}{25} e^{-t^2/25}, \quad t \geq 0. \end{aligned}$$

Also, $f(t) = 0$ for $t < 0$.

Step 3: Compute the interval probability. Using the CDF,

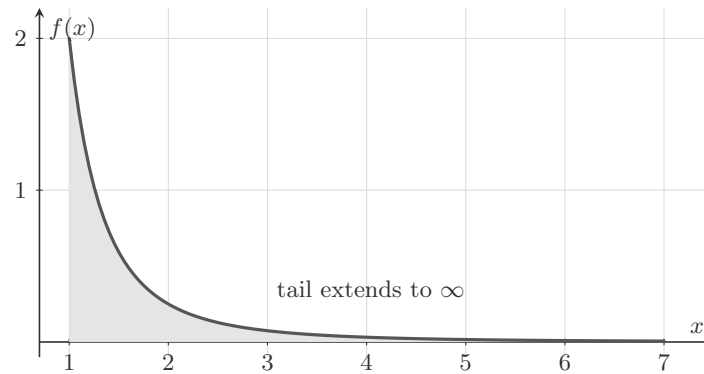
$$\begin{aligned} P(3 < X < 4) &= F(4) - F(3) \\ &= \left(1 - e^{-16/25} \right) - \left(1 - e^{-9/25} \right) \\ &= e^{-9/25} - e^{-16/25}. \end{aligned}$$

$$P(3 < X < 4) = e^{-9/25} - e^{-16/25}$$

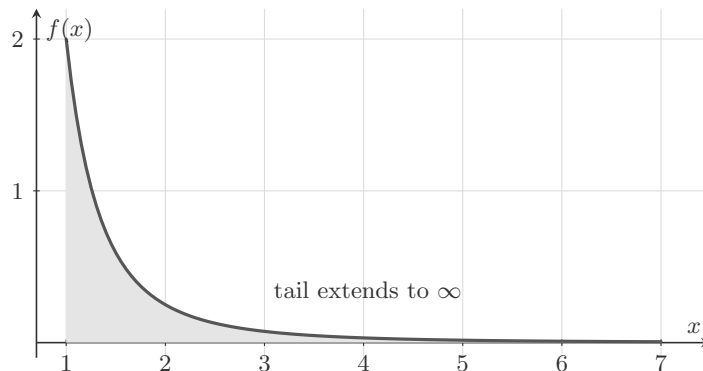
The same result follows by integrating $f(t)$ from 3 to 4.

Question 9

Choose c so $f(x) = c/x^3$, $x \geq 1$, is a density. Determine whether $E[X]$ and $E[X^2]$ are finite.



Question 9 – Solution



Step 1: Normalize the density.

$$\begin{aligned} 1 &= c \int_1^{\infty} x^{-3} dx \\ &= c \left[-\frac{1}{2x^2} \right]_1^{\infty} = \frac{c}{2}. \end{aligned}$$

Thus

$$\boxed{c = 2}.$$

Step 2: Test the first moment.

$$\begin{aligned} E[X] &= \int_1^{\infty} x \frac{2}{x^3} dx = 2 \int_1^{\infty} x^{-2} dx \\ &= 2 \left[-\frac{1}{x} \right]_1^{\infty} = 2. \end{aligned}$$

Therefore, the mean is finite.

Step 3: Test the second moment.

$$E[X^2] = \int_1^{\infty} x^2 \frac{2}{x^3} dx = 2 \int_1^{\infty} \frac{dx}{x}.$$

For an upper cutoff b ,

$$2 \int_1^b \frac{dx}{x} = 2 \ln b \rightarrow \infty.$$

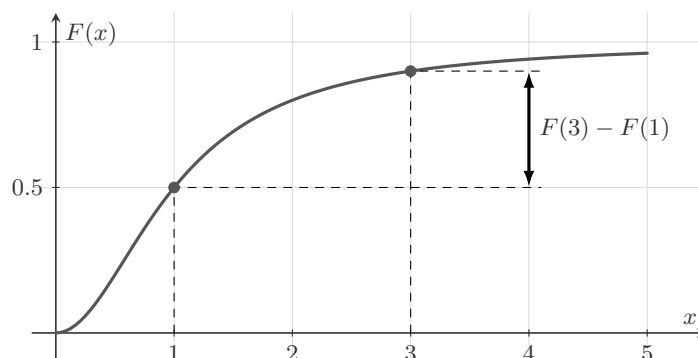
Thus the second moment diverges.

$$\boxed{E[X] = 2, \quad E[X^2] = \infty}$$

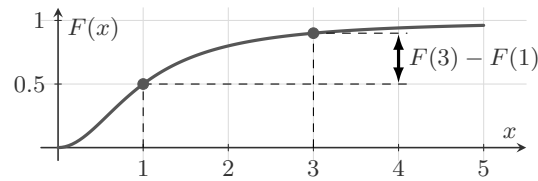
Consequently, $\text{Var}(X)$ is also infinite.

Question 10

A continuous random variable has CDF $F(x) = 0$ for $x < 0$, $F(x) = x^2/(1+x^2)$ for $x \geq 0$. Find the density and $P(1 < X < 3)$.



Question 10 – Solution



Step 1: Differentiate the CDF on each interval. For $x < 0$, $F(x) = 0$, so $f(x) = 0$. For $x > 0$,

$$\begin{aligned} f(x) &= \frac{d}{dx} \left(\frac{x^2}{1+x^2} \right) \\ &= \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} \\ &= \frac{2x}{(1+x^2)^2}. \end{aligned}$$

The CDF is continuous at $x = 0$, so there is no point mass there. Hence

$$f(x) = \begin{cases} 0, & x < 0, \\ \frac{2x}{(1+x^2)^2}, & x \geq 0. \end{cases}$$

Step 2: Use the CDF to compute the interval probability. For a continuous random variable,

$$P(1 < X < 3) = F(3) - F(1).$$

Now

$$F(3) = \frac{3^2}{1+3^2} = \frac{9}{10}, \quad F(1) = \frac{1^2}{1+1^2} = \frac{1}{2}.$$

Step 3: Subtract.

$$P(1 < X < 3) = \frac{9}{10} - \frac{1}{2} = \frac{9}{10} - \frac{5}{10} = \frac{4}{10} = \frac{2}{5}.$$

$$P(1 < X < 3) = \frac{2}{5}$$