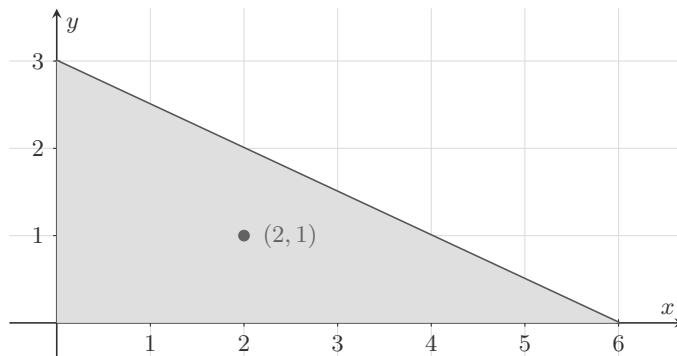


Question 1

A triangular sign occupies $x \geq 0, y \geq 0, x + 2y \leq 6$. Find its centroid using integrals.



Question 1 – Solution



Step 1: Write the upper boundary as a function of x . From $x + 2y = 6$,

$$y = f(x) = 3 - \frac{x}{2}, \quad 0 \leq x \leq 6.$$

Step 2: Find the area (and hence the mass for unit density).

$$\begin{aligned} A &= \int_0^6 f(x) dx = \int_0^6 \left(3 - \frac{x}{2}\right) dx \\ &= \left[3x - \frac{x^2}{4}\right]_0^6 = 18 - 9 = 9. \end{aligned}$$

Step 3: Find the moment about the y -axis.

$$\begin{aligned} M_y &= \int_0^6 x f(x) dx = \int_0^6 \left(3x - \frac{x^2}{2}\right) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{6}\right]_0^6 = 54 - 36 = 18. \end{aligned}$$

Step 4: Find the moment about the x -axis. Each vertical strip has centroid height $f(x)/2$, so

$$M_x = \frac{1}{2} \int_0^6 f(x)^2 dx = \frac{1}{2} \int_0^6 \left(3 - \frac{x}{2}\right)^2 dx = 9.$$

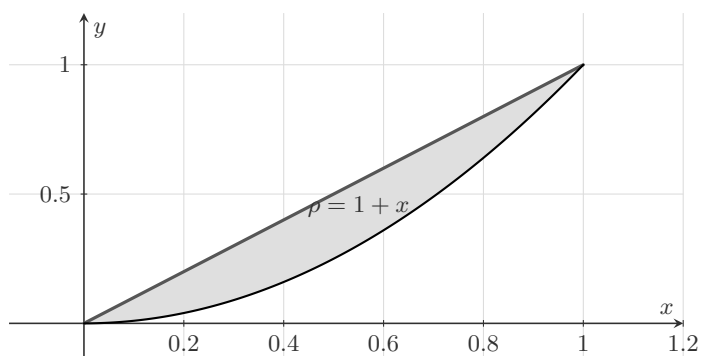
Step 5: Divide each moment by the area.

$$\bar{x} = \frac{M_y}{A} = \frac{18}{9} = 2, \quad \bar{y} = \frac{M_x}{A} = \frac{9}{9} = 1.$$

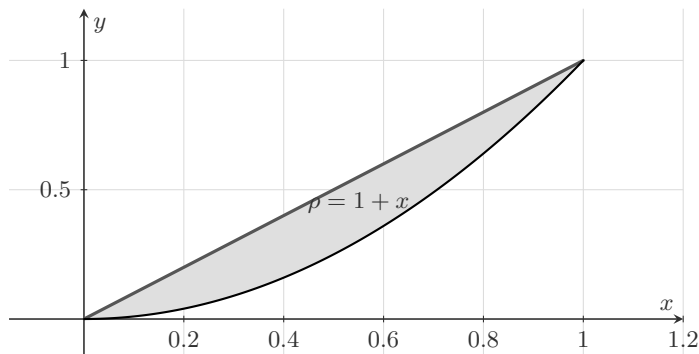
$$\boxed{(\bar{x}, \bar{y}) = (2, 1)}$$

Question 2

A lamina lies between $y = x$ and $y = x^2$, $0 \leq x \leq 1$, with density $\rho(x) = 1 + x$. Find its mass and $\bar{x}$.



Question 2 – Solution



Step 1: Identify the strip dimensions. On $0 \leq x \leq 1$, the upper curve is $y = x$ and the lower curve is $y = x^2$. Thus

$$dA = (x - x^2) dx, \quad dm = \rho(x) dA = (1 + x)(x - x^2) dx.$$

Step 2: Find the mass. Since $(1 + x)(x - x^2) = x - x^3$,

$$m = \int_0^1 (x - x^3) dx = \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{1}{4}.$$

Step 3: Find the moment about the y -axis. The moment arm is x , so

$$\begin{aligned} M_y &= \int_0^1 x dm = \int_0^1 x(x - x^3) dx \\ &= \int_0^1 (x^2 - x^4) dx = \left[\frac{x^3}{3} - \frac{x^5}{5} \right]_0^1 = \frac{2}{15}. \end{aligned}$$

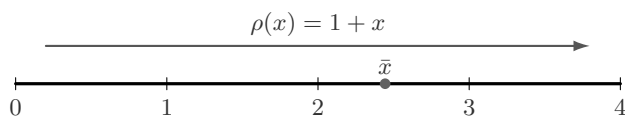
Step 4: Compute the x -coordinate of the center of mass.

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{2}{15}}{\frac{1}{4}} = \frac{8}{15}.$$

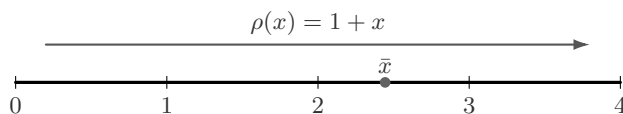
$m = \frac{1}{4}, \quad \bar{x} = \frac{8}{15}$

Question 3

A 4-m rod on $0 \leq x \leq 4$ has density $\rho(x) = k(1 + x)$. Its mass is (12) kg. Find k and its balance point.



Question 3 – Solution



Step 1: Use the total mass to determine k .

$$\begin{aligned} 12 &= \int_0^4 \rho(x) dx = k \int_0^4 (1 + x) dx \\ &= k \left[x + \frac{x^2}{2} \right]_0^4 = k(4 + 8) = 12k. \end{aligned}$$

Therefore, $k = 1$, so $\rho(x) = 1 + x$.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_0 &= \int_0^4 x\rho(x) dx = \int_0^4 (x + x^2) dx \\ &= \left[\frac{x^2}{2} + \frac{x^3}{3} \right]_0^4 = 8 + \frac{64}{3} = \frac{88}{3}. \end{aligned}$$

Step 3: Divide the moment by the mass.

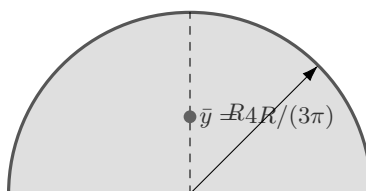
$$\bar{x} = \frac{M_0}{m} = \frac{\frac{88}{3}}{12} = \frac{22}{9} \text{ m}.$$

The balance point lies to the right of the midpoint because the density increases with x .

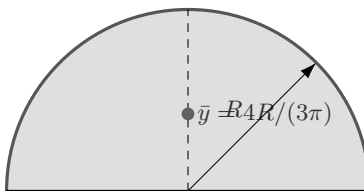
$k = 1, \quad \bar{x} = \frac{22}{9} \text{ m}$

Question 4

A uniform semicircular lamina $x^2 + y^2 \leq R^2, y \geq 0$ has $\bar{x} = 0$. Derive $\bar{y}$.



Question 4 – Solution



Step 1: Use symmetry. The lamina is symmetric about the y -axis, so

$$\bar{x} = 0.$$

Step 2: Find the area. The region is one-half of a disk of radius R :

$$A = \frac{1}{2}\pi R^2.$$

Step 3: Write the upper boundary. For $-R \leq x \leq R$,

$$f(x) = \sqrt{R^2 - x^2}.$$

A vertical strip of height $f(x)$ has centroid height $f(x)/2$.

Step 4: Find the moment about the x -axis.

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-R}^R f(x)^2 dx = \frac{1}{2} \int_{-R}^R (R^2 - x^2) dx \\ &= \frac{1}{2} \left[R^2 x - \frac{x^3}{3} \right]_{-R}^R = \frac{2R^3}{3}. \end{aligned}$$

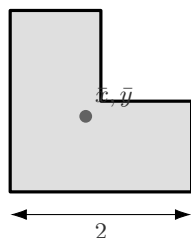
Step 5: Divide the moment by the area.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2} = \frac{4R}{3\pi}.$$

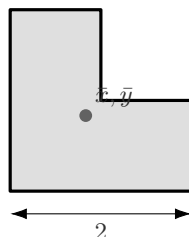
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{4R}{3\pi}\right)}$$

Question 5

A uniform square $0 \leq x, y \leq 2$ has the upper-right unit square removed. Find the centroid of the remaining L-shape.



Question 5 – Solution



Step 1: Treat the removed square as negative area. The full 2×2 square has

$$A_1 = 4, \quad (x_1, y_1) = (1, 1).$$

The removed 1×1 square has

$$A_2 = 1, \quad (x_2, y_2) = \left(\frac{3}{2}, \frac{3}{2}\right).$$

Step 2: Find the remaining area.

$$A = A_1 - A_2 = 4 - 1 = 3.$$

Step 3: Subtract the moments about the y -axis.

$$M_y = A_1 x_1 - A_2 x_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

Step 4: Subtract the moments about the x -axis.

$$M_x = A_1 y_1 - A_2 y_2 = 4(1) - 1\left(\frac{3}{2}\right) = \frac{5}{2}.$$

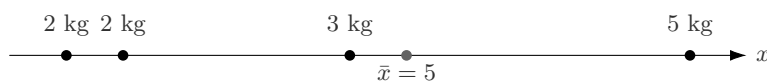
Step 5: Divide by the remaining area.

$$\bar{x} = \frac{M_y}{A} = \frac{5}{6}, \quad \bar{y} = \frac{M_x}{A} = \frac{5}{6}.$$

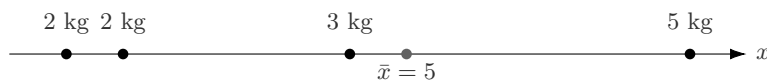
$$(\bar{x}, \bar{y}) = \left(\frac{5}{6}, \frac{5}{6}\right)$$

Question 6

Masses (2,3,5) kg sit at $x = 0, 4, 10$. Where must a (2)-kg mass be placed so the center is $x = 5$?



Question 6 – Solution



Step 1: Let a be the location of the new mass. The four masses are 2, 3, 5, and 2 kg, so the total mass is

$$m = 2 + 3 + 5 + 2 = 12 \text{ kg.}$$

Step 2: Write the center-of-mass equation. For point masses on a line,

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}.$$

Requiring $\bar{x} = 5$ gives

$$5 = \frac{2(0) + 3(4) + 5(10) + 2a}{12}.$$

Step 3: Solve for a .

$$60 = 0 + 12 + 50 + 2a,$$

$$60 = 62 + 2a,$$

$$2a = -2,$$

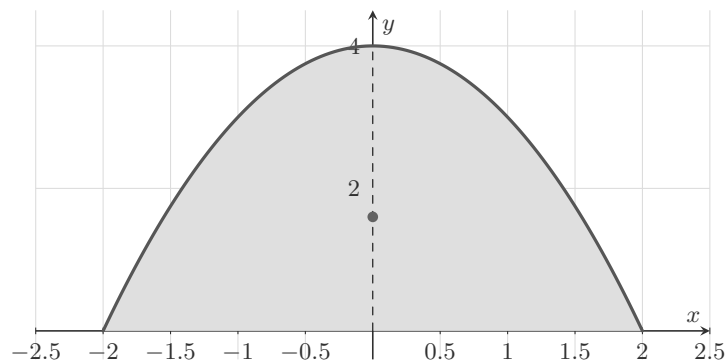
$$a = -1.$$

Step 4: Check. The total moment is $62 + 2(-1) = 60$, and $60/12 = 5$, as required.

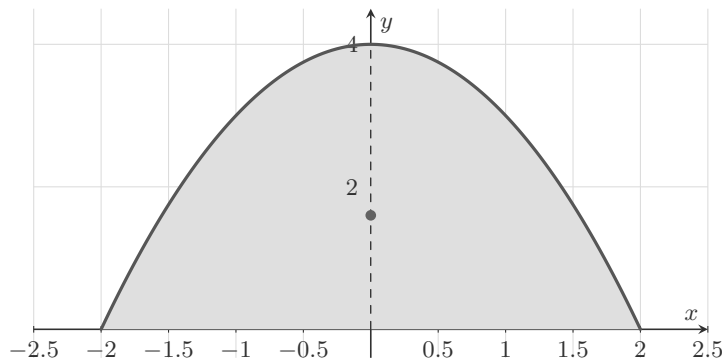
Place the 2-kg mass at $x = -1$.

Question 7

A plate under $y = 4 - x^2$, $-2 \leq x \leq 2$, has uniform density. Use symmetry before integration; find its centroid.



Question 7 – Solution



Step 1: Use symmetry before integrating. Both the region and its uniform density are symmetric about the y -axis. Therefore,

$$\bar{x} = 0.$$

Step 2: Find the area.

$$\begin{aligned} A &= \int_{-2}^2 (4 - x^2) dx = 2 \int_0^2 (4 - x^2) dx \\ &= 2 \left[4x - \frac{x^3}{3} \right]_0^2 = 2 \left(8 - \frac{8}{3} \right) = \frac{32}{3}. \end{aligned}$$

Step 3: Find the moment about the x -axis. For a vertical strip under $f(x) = 4 - x^2$,

$$\begin{aligned} M_x &= \frac{1}{2} \int_{-2}^2 f(x)^2 dx = \int_0^2 (4 - x^2)^2 dx \\ &= \int_0^2 (16 - 8x^2 + x^4) dx \\ &= \left[16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2 = \frac{256}{15}. \end{aligned}$$

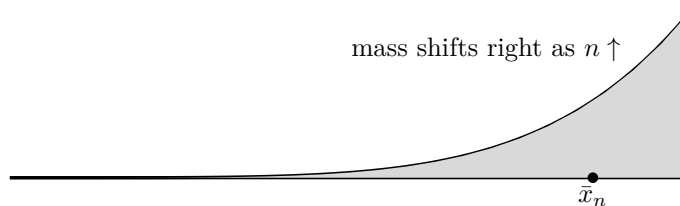
Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{A} = \frac{\frac{256}{15}}{\frac{32}{3}} = \frac{8}{5}.$$

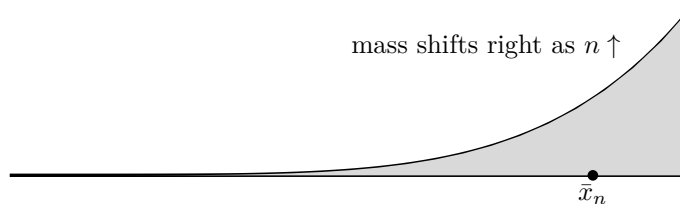
$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{8}{5} \right)}$$

Question 8

A rod on $[0, 1]$ has density $\rho_n(x) = (n+1)x^n$, for integers $n \geq 0$. Find $\bar{x}_n$ and its limiting location as $n \rightarrow \infty$.



Question 8 – Solution



Step 1: Find the total mass. For $n \geq 0$,

$$\begin{aligned} m_n &= \int_0^1 \rho_n(x) dx = \int_0^1 (n+1)x^n dx \\ &= (n+1) \left[\frac{x^{n+1}}{n+1} \right]_0^1 = 1. \end{aligned}$$

Thus the density is normalized to have total mass 1.

Step 2: Find the first moment about the origin.

$$\begin{aligned} M_n &= \int_0^1 x \rho_n(x) dx = (n+1) \int_0^1 x^{n+1} dx \\ &= (n+1) \left[\frac{x^{n+2}}{n+2} \right]_0^1 = \frac{n+1}{n+2}. \end{aligned}$$

Step 3: Compute the center of mass.

$$\bar{x}_n = \frac{M_n}{m_n} = \frac{n+1}{n+2} = 1 - \frac{1}{n+2}.$$

Step 4: Take the limit.

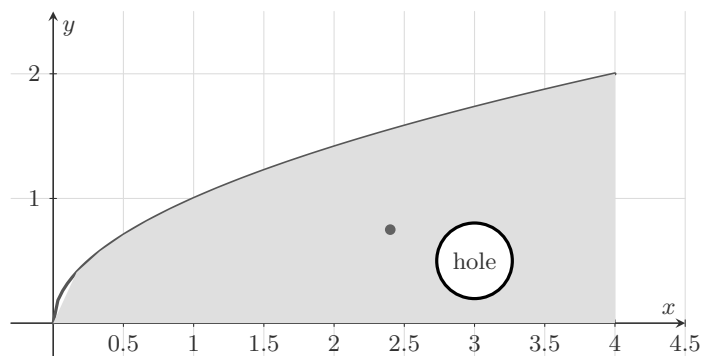
$$\lim_{n \rightarrow \infty} \bar{x}_n = 1 - \lim_{n \rightarrow \infty} \frac{1}{n+2} = 1.$$

The density becomes increasingly concentrated near the right endpoint.

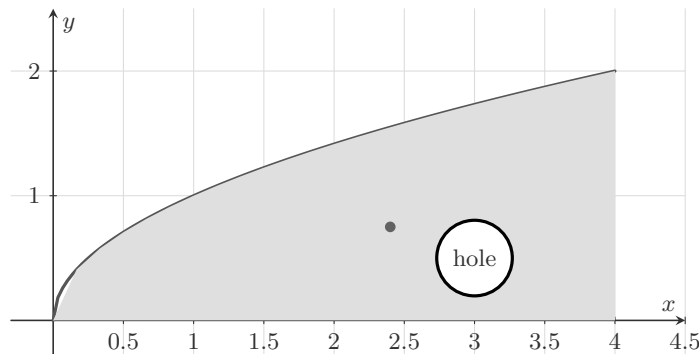
$$\boxed{\bar{x}_n = \frac{n+1}{n+2}, \quad \bar{x}_n \rightarrow 1}$$

Question 9

A lamina is bounded by $y = \sqrt{x}$, $y = 0$, $x = 4$. A circular hole of radius $(1/2)$, centered at $((3, 1/2))$, is removed. Find the new centroid using subtraction.



Question 9 – Solution



Step 1: Find the area of the original region.

$$A_0 = \int_0^4 \sqrt{x} \, dx = \left[\frac{2}{3} x^{3/2} \right]_0^4 = \frac{16}{3}.$$

Step 2: Find the original moments.

$$M_{y,0} = \int_0^4 x \sqrt{x} \, dx = \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{64}{5},$$

$$M_{x,0} = \frac{1}{2} \int_0^4 (\sqrt{x})^2 \, dx = \frac{1}{2} \int_0^4 x \, dx = 4.$$

Hence the original centroid is

$$(\bar{x}_0, \bar{y}_0) = \left(\frac{M_{y,0}}{A_0}, \frac{M_{x,0}}{A_0} \right) = \left(\frac{12}{5}, \frac{3}{4} \right).$$

Step 3: Record the hole's area and moments. The hole has radius $1/2$ and centroid $(3, 1/2)$, so

$$A_h = \pi \left(\frac{1}{2} \right)^2 = \frac{\pi}{4}, \quad M_{y,h} = 3 \left(\frac{\pi}{4} \right) = \frac{3\pi}{4}, \quad M_{x,h} = \frac{1}{2} \left(\frac{\pi}{4} \right) = \frac{\pi}{8}.$$

Step 4: Subtract the hole.

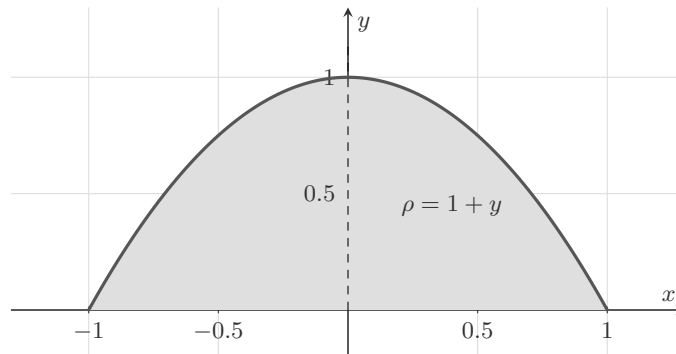
$$A = \frac{16}{3} - \frac{\pi}{4}, \quad M_y = \frac{64}{5} - \frac{3\pi}{4}, \quad M_x = 4 - \frac{\pi}{8}.$$

Step 5: Divide the remaining moments by the remaining area.

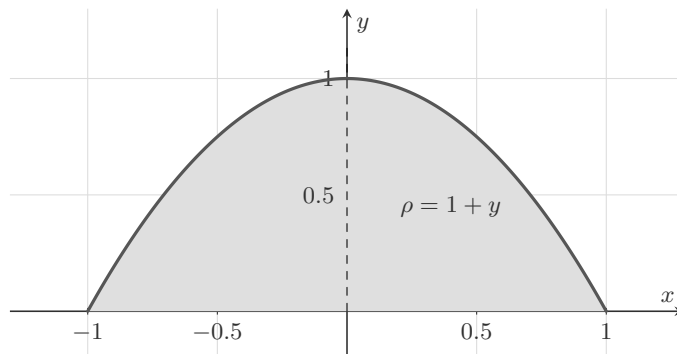
$$\boxed{\bar{x} = \frac{\frac{64}{5} - \frac{3\pi}{4}}{\frac{16}{3} - \frac{\pi}{4}}, \quad \bar{y} = \frac{4 - \frac{\pi}{8}}{\frac{16}{3} - \frac{\pi}{4}}}$$

Question 10

A symmetric lamina has density $\rho(x, y) = 1 + y$ on $-1 \leq x \leq 1, 0 \leq y \leq 1 - x^2$. State $\bar{x}$, then set up exact integrals for m and $\bar{y}$.



Question 10 – Solution



Step 1: Use symmetry. The region is symmetric about the y -axis, and $\rho(x, y) = 1 + y$ is unchanged when x is replaced by $-x$. Therefore,

$$\boxed{\bar{x} = 0}.$$

Step 2: Set up and evaluate the mass. Let $h = 1 - x^2$. Then

$$\begin{aligned} m &= \int_{-1}^1 \int_0^{1-x^2} (1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[h + \frac{h^2}{2} \right] dx \\ &= \int_{-1}^1 \left[(1-x^2) + \frac{1}{2}(1-x^2)^2 \right] dx = \frac{28}{15}. \end{aligned}$$

Step 3: Set up and evaluate the moment about the x -axis.

$$\begin{aligned} M_x &= \int_{-1}^1 \int_0^{1-x^2} y(1+y) \, dy \, dx \\ &= \int_{-1}^1 \left[\frac{h^2}{2} + \frac{h^3}{3} \right] dx \\ &= \frac{1}{2} \int_{-1}^1 (1-x^2)^2 dx + \frac{1}{3} \int_{-1}^1 (1-x^2)^3 dx \\ &= \frac{8}{15} + \frac{32}{105} = \frac{88}{105}. \end{aligned}$$

Step 4: Compute $\bar{y}$.

$$\bar{y} = \frac{M_x}{m} = \frac{\frac{88}{105}}{\frac{28}{15}} = \frac{22}{49}.$$

$$\boxed{(\bar{x}, \bar{y}) = \left(0, \frac{22}{49} \right)}$$