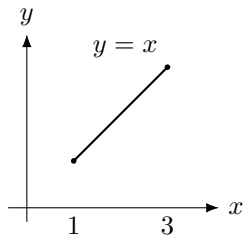


Question 1

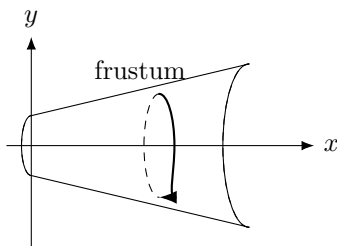
Curve: $y = x$ on $1 \leq x \leq 3$.

Axis of rotation: the x -axis.

Task: Find the lateral surface area, then verify the result with the frustum formula.



Question 1 – Solution



Step 1: Differentiate and find ds .

$$y' = 1, \quad ds = \sqrt{1 + (y')^2} dx = \sqrt{2} dx.$$

Step 2: Apply the surface-area formula.

$$\begin{aligned} S &= 2\pi \int_1^3 y \, ds \\ &= 2\pi \int_1^3 x\sqrt{2} \, dx \\ &= 2\pi\sqrt{2} \left[\frac{x^2}{2} \right]_1^3 \\ &= \pi\sqrt{2}(9 - 1) = 8\pi\sqrt{2}. \end{aligned}$$

Step 3: Verify with the frustum formula. The radii are $r_1 = 1$ and $r_2 = 3$. The slant height is

$$s = \sqrt{(3 - 1)^2 + (3 - 1)^2} = 2\sqrt{2}.$$

Therefore,

$$S = \pi(r_1 + r_2)s = \pi(1 + 3)(2\sqrt{2}) = 8\pi\sqrt{2}.$$

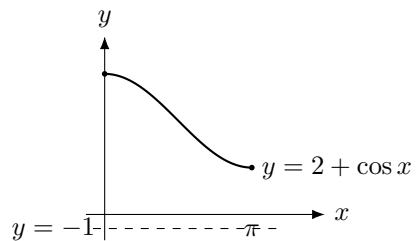
$$\boxed{S = 8\pi\sqrt{2}}$$

Question 2

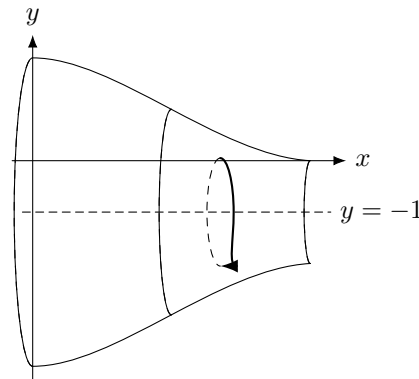
Curve: $y = 2 + \cos x$ on $0 \leq x \leq \pi$.

Axis of rotation: $y = -1$.

Task: Set up, but do not evaluate, the surface-area integral.



Question 2 – Solution



Step 1: Find the radius. The radius is the vertical distance from the curve to the axis $y = -1$:

$$r = (2 + \cos x) - (-1) = 3 + \cos x.$$

Step 2: Find the arc-length factor.

$$y' = -\sin x, \quad ds = \sqrt{1 + \sin^2 x} \, dx.$$

Step 3: Substitute into $S = 2\pi \int r \, ds$.

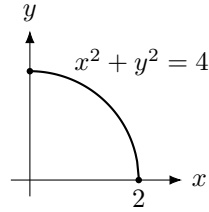
$$S = 2\pi \int_0^\pi (3 + \cos x) \sqrt{1 + \sin^2 x} \, dx$$

Question 3

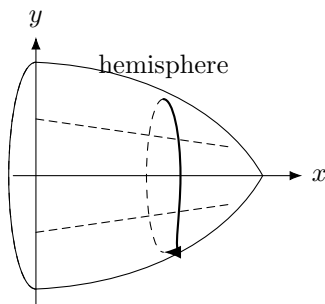
Curve: $y = \sqrt{4 - x^2}$ on $0 \leq x \leq 2$.

Axis of rotation: the x -axis.

Task: Identify the resulting surface and find its area.



Question 3 – Solution



Step 1: Identify the surface. The curve is the upper-right quarter of the circle $x^2 + y^2 = 4$. Rotating it about the x -axis produces a hemisphere of radius (2).

Step 2: Differentiate.

$$y' = -\frac{x}{\sqrt{4-x^2}}.$$

Step 3: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1+(y')^2} &= \sqrt{1 + \frac{x^2}{4-x^2}} \\ &= \sqrt{\frac{4}{4-x^2}} = \frac{2}{\sqrt{4-x^2}}.\end{aligned}$$

Step 4: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_0^2 y \sqrt{1+(y')^2} dx \\ &= 2\pi \int_0^2 \sqrt{4-x^2} \frac{2}{\sqrt{4-x^2}} dx \\ &= 4\pi \int_0^2 dx = 8\pi.\end{aligned}$$

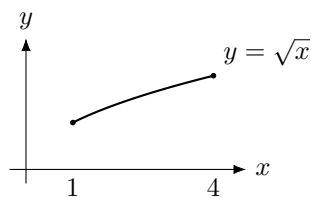
hemisphere, $S = 8\pi$

Question 4

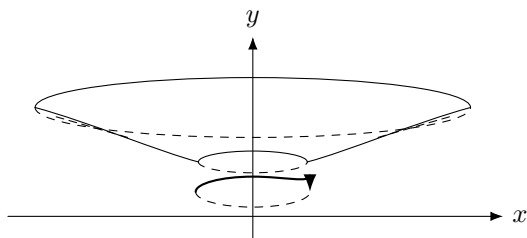
Curve: $y = \sqrt{x}$ on $1 \leq x \leq 4$.

Axis of rotation: the y -axis.

Task: Find the exact surface area.



Question 4 – Solution



Step 1: Find the radius and derivative. Rotation about the y -axis gives radius $r = x$, and

$$y' = \frac{1}{2\sqrt{x}}.$$

Step 2: Set up and simplify.

$$\begin{aligned} S &= 2\pi \int_1^4 x \sqrt{1 + \frac{1}{4x}} \, dx \\ &= \pi \int_1^4 \sqrt{x(4x+1)} \, dx. \end{aligned}$$

Step 3: Substitute $u = 8x + 1$. Then

$$dx = \frac{du}{8}, \quad \sqrt{x(4x+1)} = \frac{1}{4} \sqrt{u^2 - 1}.$$

Thus

$$S = \frac{\pi}{32} \int_9^{33} \sqrt{u^2 - 1} \, du.$$

Step 4: Integrate and return to x .

$$\int \sqrt{u^2 - 1} \, du = \frac{1}{2} \left(u \sqrt{u^2 - 1} - \ln(u + \sqrt{u^2 - 1}) \right).$$

Therefore,

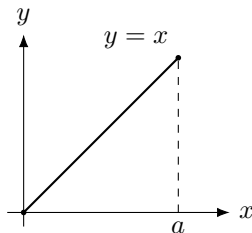
$$S = \pi \left[\frac{(8x+1)\sqrt{x(4x+1)}}{16} - \frac{1}{64} \ln \left(8x+1 + 4\sqrt{x(4x+1)} \right) \right]_1^4$$

Question 5

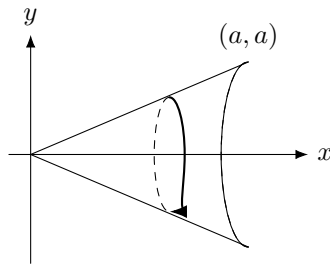
Curve: $y = x$ on $0 \leq x \leq a$, where $a > 0$.

Axis of rotation: the x -axis.

Task: Find a so that the generated surface area is $9\pi\sqrt{2}$.



Question 5 – Solution



Step 1: Find the arc-length factor.

$$y' = 1, \quad \sqrt{1 + (y')^2} = \sqrt{2}.$$

Step 2: Write and evaluate the area.

$$\begin{aligned} S &= 2\pi \int_0^a y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_0^a x \sqrt{2} dx \\ &= 2\pi \sqrt{2} \left[\frac{x^2}{2} \right]_0^a = \pi \sqrt{2} a^2. \end{aligned}$$

Step 3: Set the area equal to $9\pi\sqrt{2}$ and solve.

$$\begin{aligned} \pi \sqrt{2} a^2 &= 9\pi \sqrt{2}, \\ a^2 &= 9, \\ a &= 3 \end{aligned}$$

because $a > 0$.

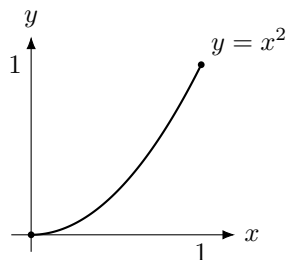
$$\boxed{a = 3}$$

Question 6

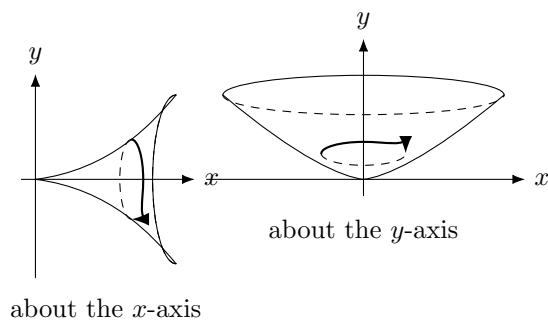
Curve: $y = x^2$ on $0 \leq x \leq 1$.

Axes of rotation: first the x -axis, then the y -axis.

Task: Write both surface-area integrals and determine which is larger without evaluating either integral.



Question 6 – Solution



Step 1: Find the common arc-length factor.

$$y' = 2x, \quad ds = \sqrt{1 + 4x^2} \, dx.$$

Step 2: Rotate about the x -axis. The radius is $y = x^2$:

$$S_x = 2\pi \int_0^1 x^2 \sqrt{1 + 4x^2} \, dx.$$

Step 3: Rotate about the y -axis. The radius is x :

$$S_y = 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx.$$

Step 4: Compare the integrands. For $0 < x < 1$,

$$x > x^2.$$

Multiplying by the positive common factor $2\pi\sqrt{1 + 4x^2}$ preserves the inequality. Therefore,

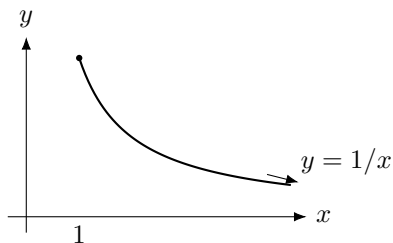
$$\boxed{S_y > S_x}.$$

Question 7

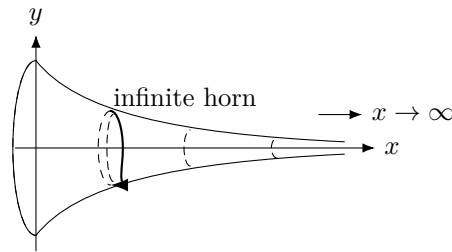
Curve: $y = 1/x$ on $x \geq 1$.

Axis of rotation: the x -axis.

Task: Determine whether the generated surface has finite area.



Question 7 – Solution



Step 1: Differentiate and set up the improper integral.

$$y' = -\frac{1}{x^2}.$$

Thus

$$S = 2\pi \int_1^{\infty} \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} dx.$$

Step 2: Find a lower comparison. Since

$$\sqrt{1 + x^{-4}} > 1,$$

we have

$$\frac{1}{x} \sqrt{1 + x^{-4}} > \frac{1}{x}.$$

Step 3: Test the comparison integral.

$$\int_1^{\infty} \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

By direct comparison, the surface-area integral diverges.

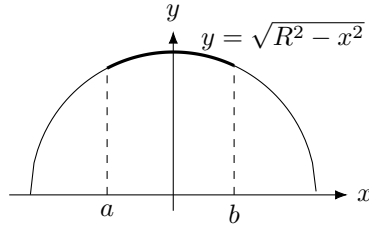
$S = \infty$ (infinite surface area)

Question 8

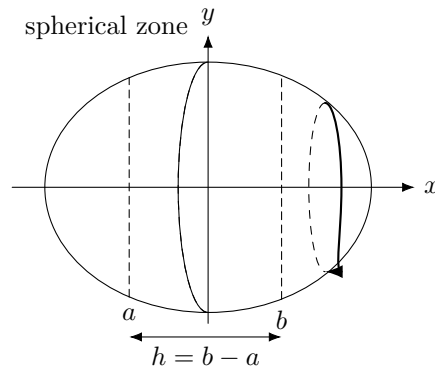
Curve: $y = \sqrt{R^2 - x^2}$ on $a \leq x \leq b$, where $R > 0$ and $-R \leq a < b \leq R$.

Axis of rotation: the x -axis.

Task: For fixed R , show that the surface area depends only on the zone height $h = b - a$.



Question 8 – Solution



Step 1: Differentiate.

$$y' = -\frac{x}{\sqrt{R^2 - x^2}}.$$

Step 2: Simplify the arc-length factor.

$$\begin{aligned}\sqrt{1 + (y')^2} &= \sqrt{1 + \frac{x^2}{R^2 - x^2}} \\ &= \sqrt{\frac{R^2}{R^2 - x^2}} = \frac{R}{\sqrt{R^2 - x^2}}.\end{aligned}$$

Step 3: Calculate the surface area.

$$\begin{aligned}S &= 2\pi \int_a^b y \sqrt{1 + (y')^2} dx \\ &= 2\pi \int_a^b \sqrt{R^2 - x^2} \frac{R}{\sqrt{R^2 - x^2}} dx \\ &= 2\pi R \int_a^b dx \\ &= 2\pi R(b - a) = 2\pi Rh.\end{aligned}$$

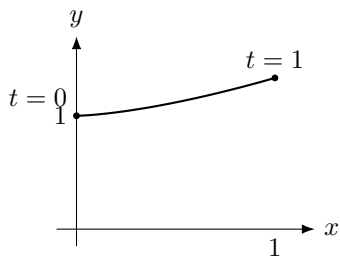
$$\boxed{S = 2\pi R(b - a) = 2\pi Rh}$$

Question 9

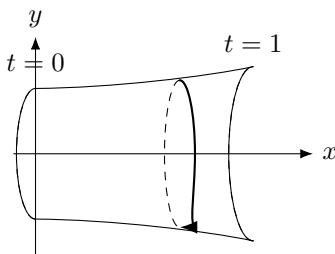
Parametric curve: $x = t^2$, $y = 1 + t^3/3$, for $0 \leq t \leq 1$.

Axis of rotation: the x -axis.

Task: Set up, but do not evaluate, the surface-area integral.



Question 9 – Solution



Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = t^2.$$

Step 2: Find the parametric arc-length factor.

$$\begin{aligned} ds &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\ &= \sqrt{4t^2 + t^4} dt \\ &= t\sqrt{4 + t^2} dt \end{aligned}$$

because $t \geq 0$.

Step 3: Use radius $y = 1 + t^3/3$.

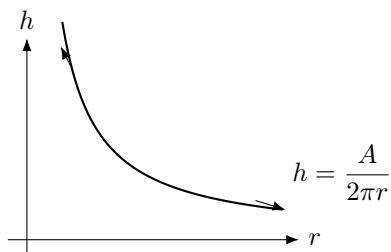
$$\begin{aligned} S &= 2\pi \int_0^1 y \, ds \\ &= 2\pi \int_0^1 \left(1 + \frac{t^3}{3}\right) t\sqrt{4 + t^2} \, dt \\ &= 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt. \end{aligned}$$

$$\boxed{S = 2\pi \int_0^1 \left(t + \frac{t^4}{3}\right) \sqrt{4 + t^2} \, dt}$$

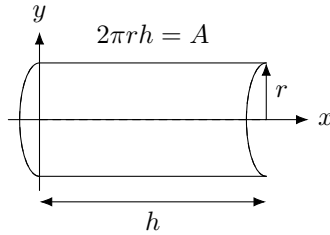
Question 10

Given: A cylinder has radius r , height h , and fixed lateral area A .

Tasks: Solve for h , find the total area including both circular ends, and then analyze the limit as $r \rightarrow 0^+$.



Question 10 – Solution



Step 1: Solve the lateral-area equation for h .

$$2\pi rh = A,$$
$$h = \frac{A}{2\pi r}.$$

Step 2: Add the two circular ends. Each end has area πr^2 , so

$$S_{\text{total}} = A + 2\pi r^2.$$

Step 3: Take the limit as $r \rightarrow 0^+$.

$$\lim_{r \rightarrow 0^+} S_{\text{total}} = \lim_{r \rightarrow 0^+} (A + 2\pi r^2) = A,$$

$$\lim_{r \rightarrow 0^+} h = \lim_{r \rightarrow 0^+} \frac{A}{2\pi r} = \infty.$$

$$\boxed{S_{\text{total}} \rightarrow A, \quad h \rightarrow \infty}$$