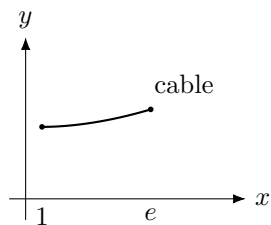


Question 1

A cable follows

$$y = \frac{x^2}{4} - \frac{1}{2} \ln x, \quad 1 \leq x \leq e.$$

Find its exact length.



Question 1 – Solution**Step 1: Differentiate.**

$$y' = \frac{x}{2} - \frac{1}{2x}.$$

Step 2: Simplify the arc-length integrand.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\frac{x}{2} - \frac{1}{2x} \right)^2 \\ &= \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} \\ &= \left(\frac{x}{2} + \frac{1}{2x} \right)^2. \end{aligned}$$

Since $x > 0$,

$$\sqrt{1 + (y')^2} = \frac{x}{2} + \frac{1}{2x}.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} L &= \int_1^e \left(\frac{x}{2} + \frac{1}{2x} \right) dx \\ &= \left[\frac{x^2}{4} + \frac{1}{2} \ln x \right]_1^e \\ &= \left(\frac{e^2}{4} + \frac{1}{2} \right) - \frac{1}{4} = \frac{e^2 + 1}{4}. \end{aligned}$$

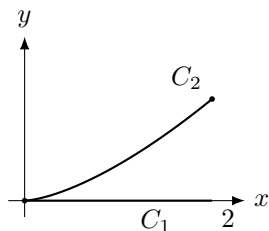
$$\boxed{L = \frac{e^2 + 1}{4}}$$

Question 2

For $0 \leq x \leq 2$, compare

$$C_1 : y = 0, \quad C_2 : y = \frac{2}{3}x^{3/2}.$$

Find $L_2 - L_1$.



Question 2 – Solution

Step 1: Find the length of C_1 . Since $y' = 0$,

$$L_1 = \int_0^2 \sqrt{1 + 0^2} \, dx = \int_0^2 1 \, dx = 2.$$

Step 2: Differentiate C_2 .

$$y' = \frac{2}{3} \cdot \frac{3}{2} x^{1/2} = \sqrt{x}.$$

Step 3: Find the length of C_2 .

$$\begin{aligned} L_2 &= \int_0^2 \sqrt{1 + (\sqrt{x})^2} \, dx \\ &= \int_0^2 \sqrt{1 + x} \, dx \\ &= \left[\frac{2}{3} (1 + x)^{3/2} \right]_0^2 \\ &= \frac{2}{3} (3\sqrt{3} - 1) = 2\sqrt{3} - \frac{2}{3}. \end{aligned}$$

Step 4: Subtract.

$$L_2 - L_1 = 2\sqrt{3} - \frac{2}{3} - 2 = 2\sqrt{3} - \frac{8}{3}.$$

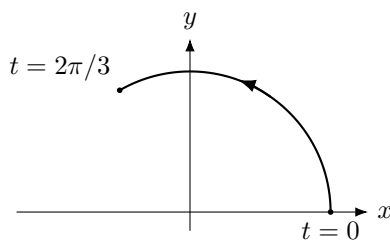
$$L_2 - L_1 = 2\sqrt{3} - \frac{8}{3}$$

Question 3

A bead moves along

$$x = 3 \cos t, \quad y = 3 \sin t, \quad 0 \leq t \leq \frac{2\pi}{3}.$$

Find the distance traveled.



Question 3 – Solution

Step 1: Differentiate both coordinates.

$$\frac{dx}{dt} = -3 \sin t, \quad \frac{dy}{dt} = 3 \cos t.$$

Step 2: Compute the speed.

$$\begin{aligned} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} &= \sqrt{9 \sin^2 t + 9 \cos^2 t} \\ &= \sqrt{9} = 3. \end{aligned}$$

Step 3: Integrate the speed.

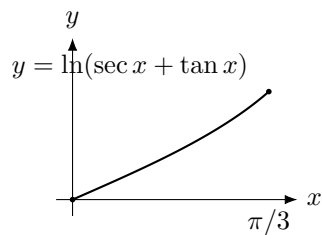
$$\begin{aligned} L &= \int_0^{2\pi/3} 3 \, dt \\ &= [3t]_0^{2\pi/3} = 2\pi. \end{aligned}$$

$$\boxed{L = 2\pi}$$

Question 4

Find the exact length of

$$y = \ln(\sec x + \tan x), \quad 0 \leq x \leq \frac{\pi}{3}.$$



Question 4 – Solution

Step 1: Differentiate.

$$y' = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x.$$

Thus

$$L = \int_0^{\pi/3} \sqrt{1 + \sec^2 x} \, dx.$$

Step 2: Substitute $u = \tan x$. Then $du = \sec^2 x \, dx$ and the bounds become 0 and $\sqrt{3}$:

$$L = \int_0^{\sqrt{3}} \frac{\sqrt{u^2 + 2}}{1 + u^2} \, du.$$

Step 3: Substitute $u = \sqrt{2} \sinh t$. Then

$$du = \sqrt{2} \cosh t \, dt, \quad \sqrt{u^2 + 2} = \sqrt{2} \cosh t.$$

The upper bound is $T = \operatorname{arsinh} \sqrt{3/2}$, and

$$\begin{aligned} L &= \int_0^T \frac{2 \cosh^2 t}{1 + 2 \sinh^2 t} \, dt \\ &= \int_0^T (1 + \operatorname{sech}(2t)) \, dt \\ &= \left[t + \frac{1}{2} \arctan(\sinh 2t) \right]_0^T. \end{aligned}$$

Since $\sinh(2T) = \sqrt{15}$,

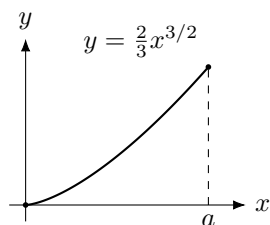
$$L = \operatorname{arsinh} \sqrt{\frac{3}{2}} + \frac{1}{2} \arctan \sqrt{15}$$

Question 5

Find $a > 0$ so that

$$y = \frac{2}{3}x^{3/2}, \quad 0 \leq x \leq a,$$

has length $14/3$.



Question 5 – Solution**Step 1: Differentiate.**

$$y' = \sqrt{x}.$$

Step 2: Write and evaluate the length.

$$\begin{aligned} L &= \int_0^a \sqrt{1+x} \, dx \\ &= \left[\frac{2}{3} (1+x)^{3/2} \right]_0^a \\ &= \frac{2}{3} \left((1+a)^{3/2} - 1 \right). \end{aligned}$$

Step 3: Set the length equal to 14/3 and solve.

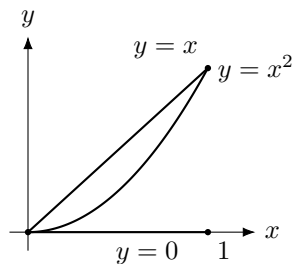
$$\begin{aligned} \frac{2}{3} \left((1+a)^{3/2} - 1 \right) &= \frac{14}{3}, \\ (1+a)^{3/2} - 1 &= 7, \\ (1+a)^{3/2} &= 8, \\ 1+a &= 8^{2/3} = 4, \\ a &= 3. \end{aligned}$$

$$\boxed{a = 3}$$

Question 6

Rank the arc lengths on $0 \leq x \leq 1$:

$$y = 0, \quad y = x, \quad y = x^2.$$



Question 6 – Solution

Step 1: Find the first two lengths.

$$L_0 = \int_0^1 \sqrt{1 + 0^2} \, dx = 1,$$

$$L_x = \int_0^1 \sqrt{1 + 1^2} \, dx = \sqrt{2} \approx 1.414.$$

Step 2: Set up the third length. For $y = x^2$, $y' = 2x$, so

$$L_{x^2} = \int_0^1 \sqrt{1 + 4x^2} \, dx.$$

Use the standard antiderivative

$$\int \sqrt{1 + 4x^2} \, dx = \frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \operatorname{arsinh}(2x).$$

Therefore,

$$\begin{aligned} L_{x^2} &= \left[\frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \operatorname{arsinh}(2x) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \operatorname{arsinh} 2 \\ &\approx 1.479. \end{aligned}$$

Step 3: Rank the values.

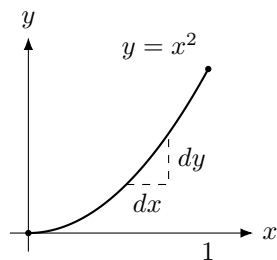
$$\boxed{L_0 < L_x < L_{x^2}}$$

Question 7

A student writes

$$L = \int_0^1 (1 + y') \, dx$$

for $y = x^2$. Explain the error and find the correct length.



Question 7 – Solution

Step 1: Correct the geometric formula. A small arc segment is the hypotenuse of a right triangle:

$$ds^2 = dx^2 + dy^2.$$

Since $dy = y' dx$,

$$ds = \sqrt{dx^2 + (y' dx)^2} = \sqrt{1 + (y')^2} dx.$$

The student's expression adds horizontal and vertical changes instead of using the Pythagorean theorem.

Step 2: Differentiate and set up the length.

$$y' = 2x, \quad L = \int_0^1 \sqrt{1 + 4x^2} dx.$$

Step 3: Evaluate.

$$\begin{aligned} L &= \left[\frac{x}{2} \sqrt{1 + 4x^2} + \frac{1}{4} \ln(2x + \sqrt{1 + 4x^2}) \right]_0^1 \\ &= \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5}). \end{aligned}$$

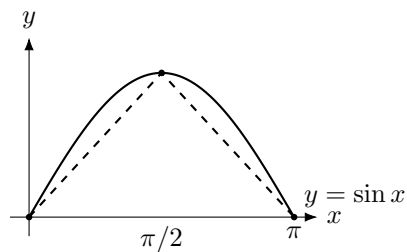
$$L = \frac{\sqrt{5}}{2} + \frac{1}{4} \ln(2 + \sqrt{5})$$

Question 8

Use two straight chords through $x = 0, \pi/2, \pi$ to estimate the length of

$$y = \sin x, \quad 0 \leq x \leq \pi.$$

Explain why the estimate is too small.



Question 8 – Solution

Step 1: List the three points.

$$(0, 0), \quad \left(\frac{\pi}{2}, 1\right), \quad (\pi, 0).$$

Step 2: Find the first chord length.

$$\begin{aligned} d_1 &= \sqrt{\left(\frac{\pi}{2} - 0\right)^2 + (1 - 0)^2} \\ &= \sqrt{\frac{\pi^2}{4} + 1}. \end{aligned}$$

By symmetry, $d_2 = d_1$. Thus

$$L_{\text{chord}} = 2d_1 = 2\sqrt{\frac{\pi^2}{4} + 1} = \sqrt{\pi^2 + 4}.$$

Step 3: Compare with the true length. The true arc length is

$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x} \, dx.$$

A curved arc between two points is longer than its straight chord. Since the sine curve is not straight on either half,

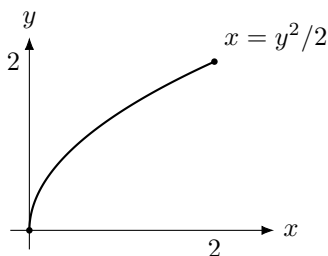
$L_{\text{chord}} = \sqrt{\pi^2 + 4}, \quad L > L_{\text{chord}}$

Question 9

For

$$x = \frac{y^2}{2}, \quad 0 \leq y \leq 2,$$

find the length using y . Then write the equivalent x -integral without evaluating it.



Question 9 – Solution

Step 1: Differentiate with respect to y .

$$\frac{dx}{dy} = y.$$

Step 2: Set up and evaluate the y -integral.

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy \\ &= \int_0^2 \sqrt{1 + y^2} dy \\ &= \left[\frac{y}{2} \sqrt{1 + y^2} + \frac{1}{2} \operatorname{arsinh} y \right]_0^2 \\ &= \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2. \end{aligned}$$

$$L = \sqrt{5} + \frac{1}{2} \operatorname{arsinh} 2$$

Step 3: Rewrite using x . Since $y = \sqrt{2x}$,

$$\frac{dy}{dx} = \frac{1}{\sqrt{2x}}.$$

The x -values run from 0 to 2, so

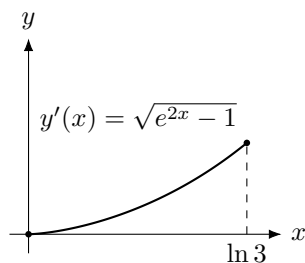
$$L = \int_0^2 \sqrt{1 + \frac{1}{2x}} dx.$$

Question 10

A curve satisfies

$$y'(x) = \sqrt{e^{2x} - 1}, \quad 0 \leq x \leq \ln 3.$$

Find its length. A formula for y is not needed.



Question 10 – Solution

Step 1: Substitute the derivative into the arc-length formula.

$$\begin{aligned} 1 + (y')^2 &= 1 + \left(\sqrt{e^{2x} - 1}\right)^2 \\ &= 1 + e^{2x} - 1 = e^{2x}. \end{aligned}$$

Step 2: Simplify the square root. Since $e^x > 0$,

$$\sqrt{1 + (y')^2} = \sqrt{e^{2x}} = e^x.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} L &= \int_0^{\ln 3} e^x \, dx \\ &= [e^x]_0^{\ln 3} \\ &= e^{\ln 3} - e^0 = 3 - 1 = 2. \end{aligned}$$

$$\boxed{L = 2}$$