

Question 1

Use direct comparison to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + \sin^2 x}.$$

Question 1 – Solution

Step 1: Find an upper comparison. Since $\sin^2 x \geq 0$,

$$x^2 + \sin^2 x \geq x^2.$$

Both sides are positive, so taking reciprocals reverses the inequality:

$$0 \leq \frac{1}{x^2 + \sin^2 x} \leq \frac{1}{x^2}.$$

Step 2: Test the comparison integral.

$$\begin{aligned} \int_1^\infty \frac{dx}{x^2} &= \lim_{b \rightarrow \infty} \left[-\frac{1}{x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(1 - \frac{1}{b} \right) = 1. \end{aligned}$$

Step 3: Apply direct comparison. The larger integral converges, so the original integral also converges.

converges

Question 2

Use limit comparison to determine convergence:

$$\int_1^{\infty} \frac{3x^2 + 1}{x^4 - 2x + 5} dx.$$

Question 2 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{3x^2}{x^4} = \frac{3}{x^2},$$

so use $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(3x^2 + 1)/(x^4 - 2x + 5)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{3x^4 + x^2}{x^4 - 2x + 5} \\ &= \lim_{x \rightarrow \infty} \frac{3 + 1/x^2}{1 - 2/x^3 + 5/x^4} = 3. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and $\int_1^\infty x^{-2} dx$ converges, the original integral converges.

converges

Question 3

Use a lower comparison to determine convergence:

$$\int_1^{\infty} \frac{2 + \sin x}{x} dx.$$

Question 3 – Solution

Step 1: Bound the numerator. Since $\sin x \geq -1$,

$$2 + \sin x \geq 1.$$

For $x \geq 1$, divide by the positive number x :

$$\frac{2 + \sin x}{x} \geq \frac{1}{x} \geq 0.$$

Step 2: Test the lower comparison.

$$\int_1^\infty \frac{dx}{x} = \lim_{b \rightarrow \infty} \ln b = \infty.$$

Step 3: Apply direct comparison. The smaller integral diverges, so the original integral diverges.

diverges

Question 4

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sqrt{x} + x}.$$

Question 4 – Solution

Step 1: Choose a comparison. Near $x = 0$, $\sqrt{x}$ is the dominant denominator term, so use

$$g(x) = \frac{1}{\sqrt{x}} = x^{-1/2}.$$

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/(\sqrt{x} + x)}{1/\sqrt{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{x} + x} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{1 + \sqrt{x}} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 x^{-1/2} dx = 2 < \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral converges.

converges

Question 5

Use comparison, not integration by parts, to determine convergence:

$$\int_2^{\infty} \frac{\ln x}{x^2} dx.$$

Question 5 – Solution

Step 1: Bound the logarithm. The logarithm grows more slowly than every positive power of x . In particular,

$$\ln x \leq \sqrt{x}$$

for all sufficiently large x .

Step 2: Divide by $x^2 > 0$.

$$0 \leq \frac{\ln x}{x^2} \leq \frac{\sqrt{x}}{x^2} = \frac{1}{x^{3/2}}.$$

Step 3: Test the upper comparison.

$$\int_2^\infty x^{-3/2} dx$$

converges because it is a p -integral with $p = 3/2 > 1$.

Step 4: Apply direct comparison. Therefore, the original integral converges.

converges

Question 6

A student uses

$$\frac{1}{\sqrt{x^2 + 1}} < \frac{1}{x}$$

to claim convergence on $[1, \infty)$. Explain the error and classify the integral.

Question 6 – Solution

Step 1: Identify the error. The comparison integral

$$\int_1^{\infty} \frac{dx}{x}$$

diverges. Being smaller than a divergent function does not prove convergence; a smaller positive function may converge or diverge.

Step 2: Use limit comparison with $g(x) = 1/x$.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/\sqrt{x^2 + 1}}{1/x} \\ &= \lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2 + 1}} \\ &= \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1 + 1/x^2}} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $0 < L < \infty$ and the harmonic integral diverges, the original integral also diverges.

diverges by limit comparison

Question 7

Use dominant powers to determine convergence:

$$\int_1^{\infty} \frac{x^3 + 4}{x^5 + x} dx.$$

Question 7 – Solution

Step 1: Choose a comparison. The dominant powers give

$$\frac{x^3}{x^5} = \frac{1}{x^2},$$

so let $g(x) = 1/x^2$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{(x^3 + 4)/(x^5 + x)}{1/x^2} \\ &= \lim_{x \rightarrow \infty} \frac{x^5 + 4x^2}{x^5 + x} \\ &= \lim_{x \rightarrow \infty} \frac{1 + 4/x^3}{1 + 1/x^4} = 1. \end{aligned}$$

Step 3: Apply limit comparison. Since $\int_1^\infty x^{-2} dx$ converges and $0 < L < \infty$, the original integral converges.

converges

Question 8

Find all real p for which the integral converges:

$$\int_1^{\infty} \frac{dx}{x^p + 1}.$$

Question 8 – Solution

Step 1: Consider $p > 0$. Compare with $g(x) = 1/x^p$:

$$\begin{aligned} L &= \lim_{x \rightarrow \infty} \frac{1/(x^p + 1)}{1/x^p} \\ &= \lim_{x \rightarrow \infty} \frac{x^p}{x^p + 1} \\ &= \lim_{x \rightarrow \infty} \frac{1}{1 + 1/x^p} = 1. \end{aligned}$$

Thus the original integral has the same behavior as the p -integral $\int_1^\infty x^{-p} dx$, which converges exactly when $p > 1$.

Step 2: Consider $p = 0$. The integrand is $1/2$, so the integral diverges.

Step 3: Consider $p < 0$. Then $x^p \rightarrow 0$, so

$$\frac{1}{x^p + 1} \rightarrow 1 \neq 0.$$

For $x \geq 1$ and $p < 0$, $x^p \leq 1$, so the integrand is at least $1/2$. Comparison with $\int_1^\infty (1/2) dx$ proves divergence.

$$\boxed{p > 1}$$

Question 9

Use limit comparison to determine convergence near $x = 0$:

$$\int_0^1 \frac{dx}{\sin x}.$$

Question 9 – Solution

Step 1: Choose a comparison. Since $\sin x \sim x$ near zero, compare with $g(x) = 1/x$.

Step 2: Compute the limit ratio.

$$\begin{aligned} L &= \lim_{x \rightarrow 0^+} \frac{1/\sin x}{1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{x}{\sin x} = 1. \end{aligned}$$

Step 3: Test the comparison integral.

$$\int_0^1 \frac{dx}{x} = \lim_{\varepsilon \rightarrow 0^+} (-\ln \varepsilon) = \infty.$$

Since $0 < L < \infty$, limit comparison shows that the original integral diverges.

diverges

Question 10

Use a two-sided estimate to determine convergence:

$$\int_1^{\infty} \frac{dx}{x^2 + 3x}.$$

Question 10 – Solution

Step 1: Bound the denominator. For $x \geq 1$, we have $x \leq x^2$, so

$$3x \leq 3x^2.$$

Adding x^2 gives

$$x^2 \leq x^2 + 3x \leq 4x^2.$$

Step 2: Take reciprocals. All terms are positive, so

$$\frac{1}{4x^2} \leq \frac{1}{x^2 + 3x} \leq \frac{1}{x^2}.$$

Step 3: Test the upper comparison.

$$\int_1^\infty \frac{dx}{x^2} = 1 < \infty.$$

Step 4: Apply direct comparison. Since the original integrand is nonnegative and bounded above by an integrable function, the integral converges.

converges