

Question 1

For which real p does the integral converge? Evaluate it when possible.

$$\int_1^{\infty} x^{-p} dx$$

Question 1 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_1^b x^{-p} dx.$$

Step 2: First suppose $p \neq 1$.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[\frac{x^{1-p}}{1-p} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \frac{b^{1-p} - 1}{1-p}. \end{aligned}$$

If $p > 1$, then $1 - p < 0$ and $b^{1-p} \rightarrow 0$, so

$$I = \frac{-1}{1-p} = \frac{1}{p-1}.$$

If $p < 1$, then $b^{1-p} \rightarrow \infty$, so the integral diverges.

Step 3: Check $p = 1$ separately.

$$\int_1^b \frac{dx}{x} = \ln b \rightarrow \infty.$$

$p > 1 : I = \frac{1}{p-1}; \quad p \leq 1 : \text{diverges}$

Question 2

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^1 x^{-2/3} dx$$

Question 2 – Solution

Step 1: Identify the improper endpoint. The integrand is undefined at $x = 0$, so replace the lower bound with $\varepsilon > 0$:

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-2/3} dx.$$

Step 2: Integrate.

$$\int x^{-2/3} dx = \frac{x^{1/3}}{1/3} = 3x^{1/3}.$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [3x^{1/3}]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (3 - 3\varepsilon^{1/3}) = 3. \end{aligned}$$

3 (convergent)

Question 3

Determine whether the integral converges. If it does, evaluate it.

$$\int_{-1}^1 \frac{dx}{\sqrt{|x|}}$$

Question 3 – Solution

Step 1: Split at the interior singularity.

$$I = \int_{-1}^0 \frac{dx}{\sqrt{|x|}} + \int_0^1 \frac{dx}{\sqrt{|x|}}.$$

The integrand is even, so the two pieces are equal:

$$I = 2 \int_0^1 x^{-1/2} dx.$$

Step 2: Write the improper limit.

$$I = 2 \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^{-1/2} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{\varepsilon \rightarrow 0^+} [2x^{1/2}]_{\varepsilon}^1 \\ &= 4 \lim_{\varepsilon \rightarrow 0^+} (1 - \sqrt{\varepsilon}) = 4. \end{aligned}$$

$I = 4 \text{ (convergent)}$

Question 4

For which real a does the integral converge? Evaluate it when possible.

$$\int_0^{\infty} e^{-ax} dx$$

Question 4 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b e^{-ax} dx.$$

Step 2: Suppose $a \neq 0$ and integrate.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{a} e^{-ax} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \frac{1 - e^{-ab}}{a}. \end{aligned}$$

If $a > 0$, then $e^{-ab} \rightarrow 0$, so $I = 1/a$. If $a < 0$, then $e^{-ab} \rightarrow \infty$, so the integral diverges.

Step 3: Check $a = 0$. Then the integrand is 1, and

$$\int_0^\infty 1 dx$$

diverges.

$a > 0 : 1/a; \quad a \leq 0 : \text{diverges}$

Question 5

Determine whether the integral converges:

$$\int_2^{\infty} \frac{dx}{x \ln x}.$$

Question 5 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x \ln x}.$$

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

The bounds become $u = \ln 2$ and $u = \ln b$, so

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \int_{\ln 2}^{\ln b} \frac{du}{u} \\ &= \lim_{b \rightarrow \infty} [\ln u]_{\ln 2}^{\ln b} \\ &= \lim_{b \rightarrow \infty} (\ln(\ln b) - \ln(\ln 2)). \end{aligned}$$

Step 3: Evaluate the limit. Since $\ln(\ln b) \rightarrow \infty$, the integral diverges.

diverges

Question 6

Evaluate the two-tailed integral:

$$\int_{-\infty}^{\infty} e^{-|x|} dx.$$

Question 6 – Solution

Step 1: Use symmetry. Since $e^{-|x|}$ is even,

$$I = 2 \int_0^{\infty} e^{-x} dx.$$

Step 2: Replace infinity with a limit.

$$I = 2 \lim_{b \rightarrow \infty} \int_0^b e^{-x} dx.$$

Step 3: Integrate and evaluate.

$$\begin{aligned} I &= 2 \lim_{b \rightarrow \infty} [-e^{-x}]_0^b \\ &= 2 \lim_{b \rightarrow \infty} (1 - e^{-b}) \\ &= 2(1 - 0) = 2. \end{aligned}$$

$$\boxed{I = 2}$$

Question 7

Determine whether the ordinary improper integral converges:

$$\int_{-1}^1 \frac{dx}{x}.$$

Question 7 – Solution

Step 1: Split at the singularity. Ordinary convergence requires both one-sided integrals to converge.

Step 2: Evaluate the left side.

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - \ln 1) = -\infty.\end{aligned}$$

Step 3: Evaluate the right side.

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \frac{dx}{x} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln 1 - \ln \varepsilon) = +\infty.\end{aligned}$$

Since the one-sided integrals diverge, the ordinary integral diverges.

diverges

The symmetric principal value is 0, but it is not an ordinary integral value.

Question 8

Evaluate the improper integral:

$$\int_0^1 \ln x \, dx.$$

Question 8 – Solution

Step 1: Replace the improper endpoint with a limit.

$$I = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln x \, dx.$$

Step 2: Integrate by parts. Let $u = \ln x$ and $dv = dx$. Then $du = dx/x$ and $v = x$:

$$\begin{aligned} \int \ln x \, dx &= x \ln x - \int x \left(\frac{1}{x} \right) dx \\ &= x \ln x - x. \end{aligned}$$

Step 3: Evaluate the limit.

$$\begin{aligned} I &= \lim_{\varepsilon \rightarrow 0^+} [x \ln x - x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln \varepsilon + \varepsilon). \end{aligned}$$

Since $\varepsilon \ln \varepsilon \rightarrow 0$ and $\varepsilon \rightarrow 0$,

$$I = -1.$$

$$\boxed{-1}$$

Question 9

Find c so that

$$f(x) = \frac{c}{1+x^2}$$

is a probability density on $(-\infty, \infty)$.

Question 9 – Solution

Step 1: Use the normalization requirement. A probability density must satisfy

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

Thus

$$1 = c \int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Step 2: Evaluate the improper integral. The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits.

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \lim_{R \rightarrow \infty} [\arctan x]_{-R}^R \\ &= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi. \end{aligned}$$

Step 3: Solve for c .

$$1 = c\pi \quad \implies \quad c = \frac{1}{\pi}.$$

Because $c > 0$, $f(x) \geq 0$ for every real x .

$c = 1/\pi$

Question 10

Determine whether the integral converges. If it does, evaluate it.

$$\int_0^{\infty} \frac{dx}{(1+x)^2}$$

Question 10 – Solution

Step 1: Replace infinity with a limit.

$$I = \lim_{b \rightarrow \infty} \int_0^b (1+x)^{-2} dx.$$

Step 2: Integrate.

$$\int (1+x)^{-2} dx = -(1+x)^{-1} = -\frac{1}{1+x}.$$

Step 3: Evaluate the bounds and limit.

$$\begin{aligned} I &= \lim_{b \rightarrow \infty} \left[-\frac{1}{1+x} \right]_0^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{1+b} + 1 \right) \\ &= 0 + 1 = 1. \end{aligned}$$

1 (convergent)
