

Question 1

Choose the shortest method and evaluate:

$$\int x e^{x^2} dx.$$

Question 1 – Solution

Step 1: Identify the inner function. The exponent is x^2 , and its derivative $2x$ is present except for a constant factor.

Step 2: Substitute $u = x^2$. Then

$$du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int x e^{x^2} \, dx &= \frac{1}{2} \int e^u \, du \\ &= \frac{1}{2} e^u + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{2} e^{x^2} + C}$$

Question 2

Choose substitution or integration by parts, then evaluate:

$$\int x \ln(1 + x^2) dx.$$

Question 2 – Solution

Step 1: Choose substitution. The expression $1 + x^2$ is inside the logarithm, and its derivative $2x$ matches the remaining factor.

Let

$$u = 1 + x^2, \quad du = 2x \, dx, \quad x \, dx = \frac{1}{2} du.$$

Then

$$I = \frac{1}{2} \int \ln u \, du.$$

Step 2: Integrate $\ln u$ by parts. Choose

$$v = \ln u, \quad dw = du.$$

Then $dv = du/u$ and $w = u$, so

$$\begin{aligned} \int \ln u \, du &= u \ln u - \int u \left(\frac{1}{u} \right) du \\ &= u \ln u - u + C. \end{aligned}$$

Step 3: Restore the factor $1/2$ and return to x .

$$\begin{aligned} I &= \frac{1}{2} (u \ln u - u) + C \\ &= \frac{1}{2} [(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C. \end{aligned}$$

$$\frac{1}{2} [(1 + x^2) \ln(1 + x^2) - (1 + x^2)] + C$$

Question 3

Use $u = x^2 + 1$ to evaluate:

$$\int \frac{x^3}{\sqrt{x^2 + 1}} dx.$$

Question 3 – Solution**Step 1: Separate one factor of x .**

$$x^3 dx = x^2(x dx).$$

Step 2: Substitute $u = x^2 + 1$. Then

$$du = 2x dx, \quad x dx = \frac{1}{2} du, \quad x^2 = u - 1.$$

Step 3: Rewrite and simplify.

$$\begin{aligned} I &= \frac{1}{2} \int \frac{u-1}{\sqrt{u}} du \\ &= \frac{1}{2} \int (u^{1/2} - u^{-1/2}) du. \end{aligned}$$

Step 4: Integrate each power.

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2} \right) + C \\ &= \frac{1}{3} u^{3/2} - u^{1/2} + C. \end{aligned}$$

Step 5: Return to x .

$$\frac{1}{3}(x^2 + 1)^{3/2} - \sqrt{x^2 + 1} + C$$

Question 4

Identify the hidden derivative and evaluate:

$$\int \frac{\ln x}{x} dx.$$

Question 4 – Solution

Step 1: Identify the inner function. Since

$$\frac{d}{dx}(\ln x) = \frac{1}{x},$$

the factor dx/x is the differential of $\ln x$.

Step 2: Substitute $u = \ln x$. Then

$$du = \frac{dx}{x}.$$

Step 3: Rewrite and integrate.

$$\begin{aligned}\int \frac{\ln x}{x} dx &= \int u du \\ &= \frac{u^2}{2} + C.\end{aligned}$$

Step 4: Return to x .

$$\frac{1}{2}(\ln x)^2 + C$$

Question 5

Recognize a product derivative and evaluate:

$$\int e^x(x+1) dx.$$

Question 5 – Solution

Step 1: Look for a product whose derivative matches the integrand. Try xe^x .

Step 2: Differentiate with the product rule.

$$\begin{aligned}\frac{d}{dx}(xe^x) &= \frac{d}{dx}(x)e^x + x\frac{d}{dx}(e^x) \\ &= e^x + xe^x \\ &= e^x(x+1).\end{aligned}$$

Step 3: Reverse the derivative.

$$\int e^x(x+1) dx = xe^x + C.$$

$xe^x + C$

Question 6

Explain why $u = x^2 + 1$ does not simplify the integral directly, then evaluate:

$$\int \frac{dx}{x^2 + 1}.$$

Question 6 – Solution

Step 1: Test the proposed substitution. If

$$u = x^2 + 1,$$

then

$$du = 2x \, dx.$$

The integral contains dx , but it does not contain the required factor $x \, dx$. Thus the substitution does not produce a direct du factor. One can express dx in terms of u on a branch, but this introduces a square root and is less convenient than the standard antiderivative.

Step 2: Recognize a standard form.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1 + x^2}.$$

Therefore,

$$\int \frac{dx}{x^2 + 1} = \arctan x + C.$$

$$\int \frac{dx}{x^2 + 1} = \arctan x + C$$

Question 7

Choose the best method and evaluate:

$$\int \frac{x^2}{x^3 + 8} dx.$$

Question 7 – Solution

Step 1: Identify the structure. The denominator is $x^3 + 8$, and its derivative is $3x^2$. The numerator contains x^2 , so substitution is the shortest method.

Step 2: Substitute $u = x^3 + 8$. Then

$$du = 3x^2 dx, \quad x^2 dx = \frac{1}{3} du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \frac{x^2}{x^3 + 8} dx &= \frac{1}{3} \int \frac{du}{u} \\ &= \frac{1}{3} \ln |u| + C. \end{aligned}$$

Step 4: Return to x .

$$\boxed{\frac{1}{3} \ln |x^3 + 8| + C}$$

Question 8

State the structural cue, then evaluate:

$$\int \sin x \cos^4 x \, dx.$$

Question 8 – Solution

Step 1: Identify the structural cue. The sine power is odd, and $\sin x \, dx$ is the differential of $-\cos x$. Keep the sine factor and use the cosine as the substitution.

Step 2: Substitute $u = \cos x$. Then

$$du = -\sin x \, dx, \quad \sin x \, dx = -du.$$

Step 3: Rewrite and integrate.

$$\begin{aligned} \int \sin x \cos^4 x \, dx &= - \int u^4 \, du \\ &= -\frac{u^5}{5} + C. \end{aligned}$$

Step 4: Return to x .

$$-\frac{1}{5} \cos^5 x + C$$

Question 9

Choose the best method for each integral. Do not evaluate.

$$\int x^2 e^x dx, \quad \int \frac{dx}{x^2 - 4}, \quad \int \sqrt{9 - x^2} dx, \quad \int \sin^2 x dx.$$

Question 9 – Solution

1. For $\int x^2 e^x dx$, differentiating x^2 repeatedly makes it simpler, while integrating e^x changes nothing. Use **integration by parts** twice.

2. For $\int dx/(x^2 - 4)$, factor

$$x^2 - 4 = (x - 2)(x + 2).$$

The denominator has distinct linear factors, so use **partial fractions**.

3. For $\int \sqrt{9 - x^2} dx$, the form $\sqrt{a^2 - x^2}$ suggests $x = a \sin \theta$. Use **trigonometric substitution** with $x = 3 \sin \theta$.

4. For $\int \sin^2 x dx$, both sine and cosine powers are even. Use the **power-reduction identity**

$$\sin^2 x = \frac{1 - \cos 2x}{2}.$$

$x^2 e^x$	integration by parts
$1/(x^2 - 4)$	partial fractions
$\sqrt{9 - x^2}$	trigonometric substitution
$\sin^2 x$	power reduction

Question 10

Find the integrand whose antiderivative is

$$F(x) = \ln(x^2 + 1) + \arctan x.$$

Then verify your result.

Question 10 – Solution

Step 1: Differentiate the logarithm using the chain rule.

$$\frac{d}{dx} \ln(x^2 + 1) = \frac{2x}{x^2 + 1}.$$

Step 2: Differentiate the inverse tangent.

$$\frac{d}{dx} \arctan x = \frac{1}{x^2 + 1}.$$

Step 3: Add and simplify.

$$\begin{aligned} F'(x) &= \frac{2x}{x^2 + 1} + \frac{1}{x^2 + 1} \\ &= \frac{2x + 1}{x^2 + 1}. \end{aligned}$$

Thus the required integrand is

$$\boxed{f(x) = \frac{2x + 1}{x^2 + 1}}$$

Step 4: Verify by integration.

$$\begin{aligned} \int \frac{2x + 1}{x^2 + 1} dx &= \int \frac{2x}{x^2 + 1} dx + \int \frac{dx}{x^2 + 1} \\ &= \ln(x^2 + 1) + \arctan x + C = F(x) + C. \end{aligned}$$