

Question 1

Complete the square and evaluate

$$\int \frac{dx}{x^2 + 6x + 13}.$$

Question 1 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 + 6x + 13 &= (x^2 + 6x + 9) + 4 \\&= (x + 3)^2 + 4.\end{aligned}$$

Step 2: Rewrite the integral.

$$I = \int \frac{dx}{(x + 3)^2 + 2^2}.$$

Step 3: Use the arctangent formula. Let $u = x + 3$, so $du = dx$:

$$\begin{aligned}I &= \int \frac{du}{u^2 + 2^2} \\&= \frac{1}{2} \arctan\left(\frac{u}{2}\right) + C \\&= \frac{1}{2} \arctan\left(\frac{x + 3}{2}\right) + C.\end{aligned}$$

$$\frac{1}{2} \arctan \frac{x + 3}{2} + C$$

Question 2

Split the numerator, then evaluate:

$$\int \frac{3x + 1}{x^2 + 2x + 5} dx.$$

Question 2 – Solution

Step 1: Split the numerator using the derivative of the denominator. Since

$$\frac{d}{dx}(x^2 + 2x + 5) = 2x + 2,$$

write

$$3x + 1 = \frac{3}{2}(2x + 2) - 2.$$

Step 2: Split the integral.

$$I = \frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx - 2 \int \frac{dx}{x^2 + 2x + 5}.$$

Step 3: Integrate the derivative term.

$$\frac{3}{2} \int \frac{2x + 2}{x^2 + 2x + 5} dx = \frac{3}{2} \ln(x^2 + 2x + 5).$$

Step 4: Complete the square in the remaining term.

$$x^2 + 2x + 5 = (x + 1)^2 + 4.$$

Thus

$$\begin{aligned} -2 \int \frac{dx}{(x + 1)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 1}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 1}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\boxed{\frac{3}{2} \ln(x^2 + 2x + 5) - \arctan \frac{x + 1}{2} + C}$$

Question 3

Evaluate

$$\int \frac{x}{x^2 + 4x + 8} dx.$$

Question 3 – Solution

Step 1: Rewrite the numerator using the derivative of the denominator.

$$x = \frac{1}{2}(2x + 4) - 2.$$

Step 2: Split the integral.

$$I = \frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx - 2 \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\frac{1}{2} \int \frac{2x + 4}{x^2 + 4x + 8} dx = \frac{1}{2} \ln(x^2 + 4x + 8).$$

Step 4: Complete the square.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\begin{aligned} -2 \int \frac{dx}{(x + 2)^2 + 2^2} &= -2 \left[\frac{1}{2} \arctan \left(\frac{x + 2}{2} \right) \right] \\ &= -\arctan \left(\frac{x + 2}{2} \right). \end{aligned}$$

Step 5: Combine the results.

$$\boxed{\frac{1}{2} \ln(x^2 + 4x + 8) - \arctan \frac{x + 2}{2} + C}$$

Question 4

Find the maximum of $f(x) = 1/(x^2 - 4x + 8)$, then evaluate

$$\int_{-\infty}^{\infty} f(x) \, dx.$$

Question 4 – Solution**Step 1: Complete the square.**

$$x^2 - 4x + 8 = (x - 2)^2 + 4.$$

Step 2: Find the maximum. The denominator is smallest when $(x - 2)^2 = 0$, or $x = 2$. Its minimum is 4, so

$$f(2) = \frac{1}{4}.$$

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 3: Write the improper integral as a limit.**

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(x - 2)^2 + 2^2} \\ &= \lim_{R \rightarrow \infty} \frac{1}{2} \left[\arctan \left(\frac{x - 2}{2} \right) \right]_{-R}^R. \end{aligned}$$

Step 4: Evaluate the limits.

$$I = \frac{1}{2} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) = \frac{\pi}{2}.$$

$\max = 1/4, \quad I = \pi/2$

Question 5

Evaluate

$$\int \frac{dx}{\sqrt{x^2 - 6x + 10}}.$$

Question 5 – Solution

Step 1: Complete the square.

$$\begin{aligned}x^2 - 6x + 10 &= (x^2 - 6x + 9) + 1 \\&= (x - 3)^2 + 1.\end{aligned}$$

Step 2: Substitute $u = x - 3$. Then $du = dx$ and

$$I = \int \frac{du}{\sqrt{u^2 + 1}}.$$

Step 3: Use the standard formula.

$$\int \frac{du}{\sqrt{u^2 + a^2}} = \ln \left| u + \sqrt{u^2 + a^2} \right| + C.$$

With $a = 1$,

$$\begin{aligned}I &= \ln \left| u + \sqrt{u^2 + 1} \right| + C \\&= \ln \left| x - 3 + \sqrt{x^2 - 6x + 10} \right| + C.\end{aligned}$$

$$\boxed{\ln |x - 3 + \sqrt{x^2 - 6x + 10}| + C}$$

Question 6

Find every k that makes $x^2 + kx + 9$ a perfect square. Then integrate

$$\frac{1}{x^2 + kx + 9}.$$

Question 6 – Solution

Step 1: Write the possible perfect squares. Since the constant term is 9, the square must be

$$(x + 3)^2 = x^2 + 6x + 9,$$

$$(x - 3)^2 = x^2 - 6x + 9.$$

Thus

$$k = 6 \quad \text{or} \quad k = -6.$$

Step 2: Integrate when $k = 6$.

$$\begin{aligned} \int \frac{dx}{x^2 + 6x + 9} &= \int \frac{dx}{(x + 3)^2} \\ &= \int (x + 3)^{-2} dx \\ &= -(x + 3)^{-1} + C = -\frac{1}{x + 3} + C. \end{aligned}$$

Step 3: Integrate when $k = -6$.

$$\begin{aligned} \int \frac{dx}{x^2 - 6x + 9} &= \int (x - 3)^{-2} dx \\ &= -(x - 3)^{-1} + C = -\frac{1}{x - 3} + C. \end{aligned}$$

$k = 6 : -\frac{1}{x + 3} + C; \quad k = -6 : -\frac{1}{x - 3} + C$

Question 7

Evaluate

$$\int \frac{x^2}{x^2 + 4} dx$$

without partial fractions.

Question 7 – Solution**Step 1: Rewrite the numerator.**

$$x^2 = (x^2 + 4) - 4.$$

Therefore,

$$\begin{aligned}\frac{x^2}{x^2 + 4} &= \frac{(x^2 + 4) - 4}{x^2 + 4} \\ &= 1 - \frac{4}{x^2 + 4}.\end{aligned}$$

Step 2: Split the integral.

$$I = \int 1 \, dx - 4 \int \frac{dx}{x^2 + 2^2}.$$

Step 3: Integrate both terms.

$$\begin{aligned}I &= x - 4 \left[\frac{1}{2} \arctan \left(\frac{x}{2} \right) \right] + C \\ &= x - 2 \arctan \left(\frac{x}{2} \right) + C.\end{aligned}$$

$$x - 2 \arctan \frac{x}{2} + C$$

Question 8

Determine the real domain and evaluate

$$\int \frac{dx}{\sqrt{6x - x^2}}.$$

Question 8 – Solution

Step 1: Complete the square.

$$\begin{aligned}6x - x^2 &= -(x^2 - 6x) \\&= -((x - 3)^2 - 9) \\&= 9 - (x - 3)^2.\end{aligned}$$

Step 2: Determine the domain. The radicand must be nonnegative:

$$\begin{aligned}9 - (x - 3)^2 &\geq 0, \\(x - 3)^2 &\leq 9, \\-3 &\leq x - 3 \leq 3, \\0 &\leq x \leq 6.\end{aligned}$$

The denominator cannot equal zero, so the integrand's domain is

$$\boxed{0 < x < 6}.$$

Step 3: Substitute $u = x - 3$, so $du = dx$:

$$I = \int \frac{du}{\sqrt{3^2 - u^2}}.$$

Step 4: Use the inverse-sine formula.

$$\begin{aligned}I &= \arcsin\left(\frac{u}{3}\right) + C \\&= \arcsin\left(\frac{x - 3}{3}\right) + C.\end{aligned}$$

$$\boxed{\arcsin \frac{x - 3}{3} + C}$$

Question 9

Split the numerator, then evaluate:

$$\int \frac{2x + 5}{x^2 + 4x + 8} dx.$$

Question 9 – Solution**Step 1: Split the numerator.**

$$2x + 5 = (2x + 4) + 1.$$

Step 2: Split the integral.

$$I = \int \frac{2x + 4}{x^2 + 4x + 8} dx + \int \frac{dx}{x^2 + 4x + 8}.$$

Step 3: Integrate the derivative term.

$$\int \frac{2x + 4}{x^2 + 4x + 8} dx = \ln(x^2 + 4x + 8).$$

Step 4: Complete the square for the remaining term.

$$x^2 + 4x + 8 = (x + 2)^2 + 4.$$

Therefore,

$$\int \frac{dx}{(x + 2)^2 + 2^2} = \frac{1}{2} \arctan \left(\frac{x + 2}{2} \right).$$

Step 5: Combine the results.

$$\boxed{\ln(x^2 + 4x + 8) + \frac{1}{2} \arctan \frac{x + 2}{2} + C}$$

Question 10

For $a > 0$, evaluate:

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + a^2}.$$

Question 10 – Solution

The antiderivative below has finite limits at both $-\infty$ and ∞ , so both one-sided improper integrals converge separately. Their sum may therefore be computed using symmetric limits. **Step 1: Write the improper integral as a limit.**

$$I = \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x^2 + a^2}.$$

Step 2: Find an antiderivative. Since $a > 0$,

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C.$$

Therefore,

$$\begin{aligned} I &= \lim_{R \rightarrow \infty} \frac{1}{a} \left[\arctan\left(\frac{x}{a}\right) \right]_{-R}^R \\ &= \frac{1}{a} \lim_{R \rightarrow \infty} \left[\arctan\left(\frac{R}{a}\right) - \arctan\left(\frac{-R}{a}\right) \right]. \end{aligned}$$

Step 3: Evaluate the limits.

$$\begin{aligned} I &= \frac{1}{a} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2} \right) \right) \\ &= \frac{\pi}{a}. \end{aligned}$$

$$\boxed{I = \frac{\pi}{a}}$$