

Question 1

Use $u = \sqrt{x}$ to evaluate

$$\int \frac{dx}{1 + \sqrt{x}}.$$

Question 1 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\int \frac{dx}{1 + \sqrt{x}} = \int \frac{2u}{1 + u} \, du.$$

Step 3: Divide the numerator.

$$\frac{u}{u+1} = \frac{(u+1)-1}{u+1} = 1 - \frac{1}{u+1}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} 2 \int \left(1 - \frac{1}{u+1} \right) du &= 2u - 2 \ln |u+1| + C \\ &= 2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C. \end{aligned}$$

$$\boxed{2\sqrt{x} - 2 \ln(1 + \sqrt{x}) + C}$$

Question 2

Use $u = x + 2$ to evaluate:

$$\int x\sqrt{x+2} \, dx.$$

Question 2 – Solution

Step 1: Substitute $u = x + 2$. Then

$$x = u - 2, \quad du = dx.$$

Step 2: Rewrite and expand the integrand.

$$\begin{aligned} \int x\sqrt{x+2} \, dx &= \int (u-2)u^{1/2} \, du \\ &= \int (u^{3/2} - 2u^{1/2}) \, du. \end{aligned}$$

Step 3: Integrate each power.

$$\begin{aligned} I &= \frac{u^{5/2}}{5/2} - 2\frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C. \end{aligned}$$

Step 4: Substitute $u = x + 2$.

$$\boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}$$

Question 3

Find the geometric area between $y = \sqrt[3]{x}$ and the x -axis on $[-1, 8]$.

Question 3 – Solution

Step 1: Split at the x -intercept. The function is negative on $[-1, 0]$ and positive on $[0, 8]$, so

$$A = - \int_{-1}^0 x^{1/3} dx + \int_0^8 x^{1/3} dx.$$

Step 2: Find an antiderivative.

$$\int x^{1/3} dx = \frac{x^{4/3}}{4/3} = \frac{3}{4}x^{4/3}.$$

Step 3: Evaluate both pieces.

$$\begin{aligned} A &= - \left[\frac{3}{4}x^{4/3} \right]_{-1}^0 + \left[\frac{3}{4}x^{4/3} \right]_0^8 \\ &= - \left(0 - \frac{3}{4} \right) + \left(\frac{3}{4} \cdot 16 - 0 \right) \\ &= \frac{3}{4} + 12 = \frac{51}{4}. \end{aligned}$$

$$\boxed{A = \frac{51}{4}}$$

Question 4

Rationalize the integrand and evaluate

$$\int \frac{dx}{\sqrt{x+1} - \sqrt{x}}.$$

Question 4 – Solution

Step 1: Multiply by the conjugate.

$$\begin{aligned}\frac{1}{\sqrt{x+1}-\sqrt{x}} &= \frac{1}{\sqrt{x+1}-\sqrt{x}} \cdot \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} \\ &= \frac{\sqrt{x+1}+\sqrt{x}}{(x+1)-x} \\ &= \sqrt{x+1}+\sqrt{x}.\end{aligned}$$

Step 2: Integrate both terms.

$$\begin{aligned}I &= \int (x+1)^{1/2} dx + \int x^{1/2} dx \\ &= \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + C \\ &= \frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C.\end{aligned}$$

$$\boxed{\frac{2}{3}(x+1)^{3/2} + \frac{2}{3}x^{3/2} + C}$$

Question 5

Evaluate

$$\int \frac{\sqrt{x}}{1+x} dx.$$

Question 5 – Solution

Step 1: Substitute $u = \sqrt{x}$. Then

$$x = u^2, \quad dx = 2u \, du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} \int \frac{\sqrt{x}}{1+x} dx &= \int \frac{u}{1+u^2} (2u \, du) \\ &= 2 \int \frac{u^2}{1+u^2} du. \end{aligned}$$

Step 3: Divide the numerator.

$$\frac{u^2}{1+u^2} = \frac{(1+u^2) - 1}{1+u^2} = 1 - \frac{1}{1+u^2}.$$

Step 4: Integrate and return to x .

$$\begin{aligned} I &= 2 \int \left(1 - \frac{1}{1+u^2} \right) du \\ &= 2u - 2 \arctan u + C \\ &= 2\sqrt{x} - 2 \arctan \sqrt{x} + C. \end{aligned}$$

$2\sqrt{x} - 2 \arctan \sqrt{x} + C$

Question 6

Find $a > 0$ so

$$\int_0^a \sqrt{x} \, dx = 18.$$

Question 6 – Solution

Step 1: Evaluate the definite integral.

$$\begin{aligned}\int_0^a \sqrt{x} \, dx &= \int_0^a x^{1/2} \, dx \\ &= \left[\frac{x^{3/2}}{3/2} \right]_0^a \\ &= \left[\frac{2}{3} x^{3/2} \right]_0^a = \frac{2}{3} a^{3/2}.\end{aligned}$$

Step 2: Set the result equal to 18 and solve.

$$\begin{aligned}\frac{2}{3} a^{3/2} &= 18, \\ a^{3/2} &= 27, \\ \left(a^{3/2} \right)^{2/3} &= 27^{2/3}, \\ a &= (\sqrt[3]{27})^2 = 3^2 = 9.\end{aligned}$$

$a = 9$

Question 7

Evaluate

$$\int \frac{dx}{x\sqrt[3]{x+1}}$$

using $u = \sqrt[3]{x+1}$.

Question 7 – Solution

Step 1: Substitute $u = \sqrt[3]{x+1}$. Then

$$u^3 = x + 1, \quad x = u^3 - 1, \quad dx = 3u^2 du.$$

Step 2: Rewrite and factor.

$$I = \int \frac{3u^2 du}{(u^3 - 1)u} = 3 \int \frac{u}{u^3 - 1} du,$$
$$u^3 - 1 = (u - 1)(u^2 + u + 1).$$

Step 3: Use partial fractions.

$$\frac{3u}{u^3 - 1} = \frac{A}{u - 1} + \frac{Bu + C}{u^2 + u + 1}.$$

After clearing denominators,

$$3u = A(u^2 + u + 1) + (Bu + C)(u - 1).$$

Matching coefficients gives

$$A + B = 0, \quad A - B + C = 3, \quad A - C = 0,$$

so $A = 1$, $B = -1$, and $C = 1$. Therefore,

$$\frac{3u}{u^3 - 1} = \frac{1}{u - 1} + \frac{-u + 1}{u^2 + u + 1}.$$

Step 4: Prepare the quadratic term.

$$-u + 1 = -\frac{1}{2}(2u + 1) + \frac{3}{2}.$$

Thus

$$I = \int \frac{du}{u - 1} - \frac{1}{2} \int \frac{2u + 1}{u^2 + u + 1} du + \frac{3}{2} \int \frac{du}{u^2 + u + 1}.$$

Complete the square:

$$u^2 + u + 1 = \left(u + \frac{1}{2}\right)^2 + \frac{3}{4}.$$

Step 5: Integrate and return to x .

$$I = \ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C,$$

where $u = \sqrt[3]{x+1}$.

$$\ln|u - 1| - \frac{1}{2} \ln(u^2 + u + 1) + \sqrt{3} \arctan\left(\frac{2u + 1}{\sqrt{3}}\right) + C$$

Question 8

Evaluate

$$\int_0^1 \frac{dx}{\sqrt{x} + \sqrt[3]{x}}$$

using a substitution that clears both radicals.

Question 8 – Solution

Step 1: Choose a power that clears both radicals. Since $\text{lcm}(2, 3) = 6$, let $x = u^6$. Then

$$\sqrt{x} = u^3, \quad \sqrt[3]{x} = u^2, \quad dx = 6u^5 du.$$

The bounds remain 0 and 1 because $u = x^{1/6}$. **Step 2: Rewrite and simplify.**

$$\begin{aligned} I &= 6 \int_0^1 \frac{u^5}{u^3 + u^2} du \\ &= 6 \int_0^1 \frac{u^5}{u^2(u+1)} du = 6 \int_0^1 \frac{u^3}{u+1} du. \end{aligned}$$

Step 3: Divide the polynomials.

$$\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}.$$

Step 4: Integrate and evaluate the bounds.

$$\begin{aligned} I &= 6 \int_0^1 \left(u^2 - u + 1 - \frac{1}{u+1} \right) du \\ &= 6 \left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1) \right]_0^1 \\ &= 6 \left(\frac{1}{3} - \frac{1}{2} + 1 - \ln 2 \right) \\ &= 6 \left(\frac{5}{6} - \ln 2 \right) = 5 - 6 \ln 2. \end{aligned}$$

$$\boxed{I = 5 - 6 \ln 2}$$

Question 9

Differentiate $F(x)$, then identify the substitution used to integrate $F'(x)$:

$$F(x) = \frac{2}{3}(1 + x^2)^{3/2}.$$

Question 9 – Solution

Step 1: Differentiate with the chain rule.

$$\begin{aligned} F'(x) &= \frac{2}{3} \cdot \frac{3}{2} (1+x^2)^{1/2} \cdot (2x) \\ &= 2x\sqrt{1+x^2}. \end{aligned}$$

Step 2: Reverse the process by integration.

$$\int 2x\sqrt{1+x^2} \, dx.$$

Let

$$u = 1 + x^2, \quad du = 2x \, dx.$$

Then

$$\begin{aligned} \int 2x\sqrt{1+x^2} \, dx &= \int u^{1/2} \, du \\ &= \frac{u^{3/2}}{3/2} + C \\ &= \frac{2}{3}(1+x^2)^{3/2} + C. \end{aligned}$$

$$\boxed{\int 2x\sqrt{1+x^2} \, dx = F(x) + C}$$

Question 10

Use one substitution to evaluate:

$$\int \frac{x+1}{\sqrt{x^2+2x+5}} dx.$$

Question 10 – Solution

Step 1: Substitute the expression inside the square root.

$$u = x^2 + 2x + 5.$$

Differentiate:

$$du = (2x + 2) dx = 2(x + 1) dx, \quad (x + 1) dx = \frac{1}{2} du.$$

Step 2: Rewrite the integral.

$$\begin{aligned} I &= \int \frac{x + 1}{\sqrt{x^2 + 2x + 5}} dx \\ &= \frac{1}{2} \int u^{-1/2} du. \end{aligned}$$

Step 3: Integrate and return to x .

$$\begin{aligned} I &= \frac{1}{2} \left(\frac{u^{1/2}}{1/2} \right) + C \\ &= u^{1/2} + C \\ &= \sqrt{x^2 + 2x + 5} + C. \end{aligned}$$

$\sqrt{x^2 + 2x + 5} + C$