

Question 1

Find A and B , then evaluate the integral:

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

$$\int \frac{7x+1}{(x-1)(x+2)} dx.$$

Question 1 – Solution

Step 1: Set up the decomposition.

$$\frac{7x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Step 2: Clear the denominators. Multiply by $(x-1)(x+2)$:

$$7x+1 = A(x+2) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad 8 &= 3A \quad \implies \quad A = \frac{8}{3}, \\ x=-2: \quad -13 &= -3B \quad \implies \quad B = \frac{13}{3}. \end{aligned}$$

Step 4: Substitute and integrate each term.

$$\begin{aligned} \int \frac{7x+1}{(x-1)(x+2)} dx &= \int \left(\frac{8/3}{x-1} + \frac{13/3}{x+2} \right) dx \\ &= \frac{8}{3} \int \frac{dx}{x-1} + \frac{13}{3} \int \frac{dx}{x+2} \\ &= \boxed{\frac{8}{3} \ln|x-1| + \frac{13}{3} \ln|x+2| + C}. \end{aligned}$$

Question 2

Use polynomial division, then evaluate:

$$\int \frac{x^3 + 1}{x^2 - 1} dx.$$

State the excluded values of x .

Question 2 – Solution

Step 1: Divide the polynomials. Since $x^3/x^2 = x$,

$$\begin{aligned}x(x^2 - 1) &= x^3 - x, \\(x^3 + 1) - (x^3 - x) &= x + 1.\end{aligned}$$

Thus $x^3 + 1 = x(x^2 - 1) + (x + 1)$, so

$$\frac{x^3 + 1}{x^2 - 1} = x + \frac{x + 1}{x^2 - 1}.$$

Step 2: Factor and simplify.

$$\begin{aligned}\frac{x^3 + 1}{x^2 - 1} &= x + \frac{x + 1}{(x - 1)(x + 1)} \\&= x + \frac{1}{x - 1},\end{aligned}$$

because the original denominator is zero at both $x = -1$ and $x = 1$.

Step 3: Integrate term by term.

$$\begin{aligned}\int \frac{x^3 + 1}{x^2 - 1} dx &= \int x dx + \int \frac{dx}{x - 1} \\&= \boxed{\frac{x^2}{2} + \ln |x - 1| + C}, \quad x \neq \pm 1.\end{aligned}$$

Question 3

Decompose into partial fractions. Include every power of $(x - 1)$:

$$\frac{2x + 3}{x(x - 1)^2}.$$

Do not integrate.

Question 3 – Solution

Step 1: Include a term for every factor and repeated power.

$$\frac{2x+3}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Step 2: Clear the denominators.

$$2x+3 = A(x-1)^2 + Bx(x-1) + Cx.$$

Step 3: Find A and C .

$$x=0: \quad 3 = A(-1)^2 \implies A=3,$$

$$x=1: \quad 5 = C(1) \implies C=5.$$

Step 4: Find B by using $x=2$.

$$\begin{aligned} 7 &= 3(1)^2 + B(2)(1) + 5(2) \\ &= 3 + 2B + 10, \\ -6 &= 2B \implies B = -3. \end{aligned}$$

Step 5: State the result.

$$\boxed{\frac{2x+3}{x(x-1)^2} = \frac{3}{x} - \frac{3}{x-1} + \frac{5}{(x-1)^2}},$$

and the requested coefficients are

$$\boxed{A=3, \quad B=-3, \quad C=5}.$$

Question 4

Use partial fractions to evaluate:

$$\int \frac{x^2 + 1}{x(x^2 + 4)} dx.$$

Question 4 – Solution

Step 1: Set up the decomposition. Use a linear numerator over the irreducible quadratic:

$$\frac{x^2 + 1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}.$$

Step 2: Clear denominators and collect like terms.

$$\begin{aligned}x^2 + 1 &= A(x^2 + 4) + x(Bx + C) \\&= (A + B)x^2 + Cx + 4A.\end{aligned}$$

Step 3: Match coefficients.

$$A + B = 1, \quad C = 0, \quad 4A = 1.$$

Therefore $A = 1/4$, $B = 1 - 1/4 = 3/4$, and $C = 0$.

Step 4: Integrate. For the second term, let $u = x^2 + 4$; then $du = 2x \, dx$.

$$\begin{aligned}I &= \frac{1}{4} \int \frac{dx}{x} + \frac{3}{4} \int \frac{x}{x^2 + 4} \, dx \\&= \frac{1}{4} \ln|x| + \frac{3}{4} \left(\frac{1}{2} \int \frac{du}{u} \right) \\&= \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C.\end{aligned}$$

Thus,

$$\boxed{I = \frac{1}{4} \ln|x| + \frac{3}{8} \ln(x^2 + 4) + C}.$$

Question 5

Find the value of k that removes the $\ln|x+1|$ term. Then evaluate:

$$\int \frac{kx+1}{x^2-1} dx.$$

Question 5 – Solution**Step 1: Set up the decomposition.**

$$\frac{kx+1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Step 2: Clear the denominators.

$$kx+1 = A(x+1) + B(x-1).$$

Step 3: Find the coefficients.

$$\begin{aligned} x=1: \quad k+1 &= 2A \implies A = \frac{k+1}{2}, \\ x=-1: \quad -k+1 &= -2B \implies B = \frac{k-1}{2}. \end{aligned}$$

Step 4: Remove the $\ln|x+1|$ term. Since integrating $B/(x+1)$ gives $B \ln|x+1|$, set $B=0$:

$$B=0 \implies \frac{k-1}{2} = 0 \implies \boxed{k=1}.$$

Step 5: Substitute $k=1$, simplify, and integrate.

$$\frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}, \quad x \neq \pm 1.$$

$$\int \frac{x+1}{x^2-1} dx = \int \frac{dx}{x-1} = \boxed{\ln|x-1| + C}.$$

Question 6

A student proposes the decomposition

$$\frac{1}{(x^2 + 1)^2} = \frac{A}{x^2 + 1} + \frac{B}{(x^2 + 1)^2}.$$

1. Why is this decomposition not useful? Give the correct general form.
2. Use $x = \tan \theta$ to evaluate:

$$\int \frac{dx}{(x^2 + 1)^2}$$

Question 6 – Solution**Part 1: Check the decomposition.**

Each power of an irreducible quadratic needs a linear numerator. The general form is

$$\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}.$$

The original fraction already has this form: $A = B = C = 0$ and $D = 1$. Therefore, decomposing it only reproduces the original integrand and does not make the integral easier.

Part 2: Evaluate the integral.

Step 1: Substitute $x = \tan \theta$. Then

$$dx = \sec^2 \theta \, d\theta, \quad x^2 + 1 = \tan^2 \theta + 1 = \sec^2 \theta.$$

Step 2: Simplify and integrate.

$$\begin{aligned} I &= \int \frac{\sec^2 \theta}{\sec^4 \theta} d\theta = \int \cos^2 \theta \, d\theta \\ &= \frac{1}{2} \int (1 + \cos 2\theta) \, d\theta \\ &= \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C. \end{aligned}$$

Step 3: Return to x .

$$\theta = \arctan x, \quad \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \frac{2x}{1 + x^2}.$$

Substitute both expressions:

$$\boxed{I = \frac{x}{2(x^2 + 1)} + \frac{1}{2} \arctan x + C}.$$

Question 7

Use partial fractions to evaluate exactly:

$$\int_0^{1/2} \frac{dx}{1-x^2}.$$

Why is this integral proper on $[0, 1/2]$?

Question 7 – Solution

Step 1: Factor and set up partial fractions.

$$1 - x^2 = (1 - x)(1 + x),$$

$$\frac{1}{1 - x^2} = \frac{A}{1 - x} + \frac{B}{1 + x}.$$

Step 2: Clear denominators and find A and B .

$$1 = A(1 + x) + B(1 - x).$$

$$x = 1 : \quad 1 = 2A \implies A = \frac{1}{2},$$

$$x = -1 : \quad 1 = 2B \implies B = \frac{1}{2}.$$

Step 3: Integrate and evaluate both bounds.

$$\begin{aligned} I &= \frac{1}{2} \int_0^{1/2} \left(\frac{1}{1 - x} + \frac{1}{1 + x} \right) dx \\ &= \frac{1}{2} [-\ln(1 - x) + \ln(1 + x)]_0^{1/2} \\ &= \frac{1}{2} \left[-\ln \frac{1}{2} + \ln \frac{3}{2} - (-\ln 1 + \ln 1) \right] \\ &= \frac{1}{2} \ln \left(\frac{3/2}{1/2} \right) = \frac{1}{2} \ln 3. \end{aligned}$$

Step 4: Check whether the integral is proper. The singularities $x = \pm 1$ are outside $[0, 1/2]$, so the integrand is continuous on the whole interval and no improper limit is needed.

$$\boxed{\int_0^{1/2} \frac{dx}{1 - x^2} = \frac{1}{2} \ln 3.}$$

Question 8

Find A , B , and C in the decomposition

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}.$$

Use $x = 0$, 2 , and -3 . State (A, B, C) .

Question 8 – Solution

Step 1: Start with the decomposition.

$$\frac{1}{x(x-2)(x+3)} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{x+3}$$

Step 2: Multiply every term by $x(x-2)(x+3)$.

$$1 = A(x-2)(x+3) + Bx(x+3) + Cx(x-2).$$

Step 3: Use values that eliminate two terms at a time.

$$\begin{array}{llll} x = 0 : & 1 = -6A & \implies & A = -\frac{1}{6}, \\ x = 2 : & 1 = 10B & \implies & B = \frac{1}{10}, \\ x = -3 : & 1 = 15C & \implies & C = \frac{1}{15}. \end{array}$$

Step 4: State the ordered triple.

$$(A, B, C) = \left(-\frac{1}{6}, \frac{1}{10}, \frac{1}{15} \right).$$

Question 9

Use partial fractions to evaluate:

$$\int \frac{dx}{x^3 - x^2} = \int \frac{dx}{x^2(x - 1)}.$$

Question 9 – Solution

Step 1: Set up partial fractions. The repeated factor x^2 requires both $1/x$ and $1/x^2$:

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Step 2: Clear the denominators.

$$1 = Ax(x-1) + B(x-1) + Cx^2.$$

Step 3: Find B and C .

$$x = 0: \quad 1 = -B \implies B = -1,$$

$$x = 1: \quad 1 = C \implies C = 1.$$

Step 4: Find A by setting $x = 2$.

$$\begin{aligned} 1 &= A(2)(1) + (-1)(1) + (1)(4) \\ &= 2A + 3, \\ -2 &= 2A \implies A = -1. \end{aligned}$$

Step 5: Substitute and integrate each term.

$$\begin{aligned} I &= \int \left(-\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} \right) dx \\ &= -\int \frac{dx}{x} - \int x^{-2} dx + \int \frac{dx}{x-1} \\ &= -\ln|x| - \frac{x^{-1}}{-1} + \ln|x-1| + C \\ &= -\ln|x| + \frac{1}{x} + \ln|x-1| + C. \end{aligned}$$

Therefore,

$$\boxed{I = -\ln|x| + \frac{1}{x} + \ln|x-1| + C}.$$

Question 10

1. Define and evaluate the Cauchy principal value:

$$\text{PV} \int_{-1}^1 \frac{dx}{x}$$

2. Does the ordinary improper integral converge? Check both sides of $x = 0$.

Question 10 – Solution

Part 1: Cauchy principal value.

Step 1: Approach zero by the same distance ε from both sides.

$$\begin{aligned}\text{PV} \int_{-1}^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} \left(\int_{-1}^{-\varepsilon} \frac{dx}{x} + \int_{\varepsilon}^1 \frac{dx}{x} \right) \\ &= \lim_{\varepsilon \rightarrow 0^+} ([\ln |x|]_{-1}^{-\varepsilon} + [\ln |x|]_{\varepsilon}^1) \\ &= \lim_{\varepsilon \rightarrow 0^+} ((\ln \varepsilon - \ln 1) + (\ln 1 - \ln \varepsilon)) \\ &= \lim_{\varepsilon \rightarrow 0^+} 0 = 0.\end{aligned}$$

$$\boxed{\text{PV} \int_{-1}^1 \frac{dx}{x} = 0}.$$

Part 2: Ordinary improper integral.

The two one-sided integrals must converge separately. On the right,

$$\begin{aligned}\int_0^1 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln x]_{\varepsilon}^1 \\ &= \lim_{\varepsilon \rightarrow 0^+} (0 - \ln \varepsilon) = +\infty.\end{aligned}$$

On the left,

$$\begin{aligned}\int_{-1}^0 \frac{dx}{x} &= \lim_{\varepsilon \rightarrow 0^+} [\ln |x|]_{-1}^{-\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} (\ln \varepsilon - 0) = -\infty.\end{aligned}$$

Therefore, the ordinary improper integral diverges. The principal value is finite only because these opposite divergences cancel in a symmetric limit.