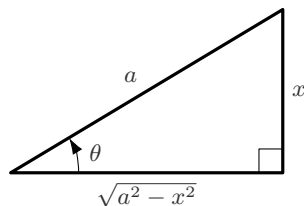


Question 1

Evaluate the integral both geometrically and analytically:

$$\int_0^a \sqrt{a^2 - x^2} \, dx, \quad a > 0.$$

Question 1 – Solution



Geometrically, $y = \sqrt{a^2 - x^2}$ is the upper semicircle $x^2 + y^2 = a^2$. On $0 \leq x \leq a$, the region is one quarter of a disk of radius a . Hence its area is

$$I = \frac{1}{4}\pi a^2.$$

For an analytic verification, let

$$x = a \sin \theta, \quad dx = a \cos \theta \, d\theta.$$

Since $0 \leq x \leq a$, the new bounds are $0 \leq \theta \leq \pi/2$. Also,

$$\begin{aligned} \sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ &= a \cos \theta, \end{aligned}$$

because $\cos \theta \geq 0$ on this interval. Therefore,

$$\begin{aligned} I &= a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta \\ &= \frac{a^2}{2} \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta \\ &= \frac{a^2}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\pi/2} = \frac{\pi a^2}{4}. \end{aligned}$$

Thus both methods give

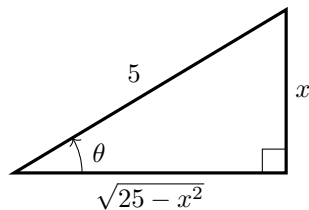
$$\boxed{\int_0^a \sqrt{a^2 - x^2} \, dx = \frac{\pi a^2}{4}}.$$

Question 2

Evaluate using a trigonometric substitution:

$$\int \frac{x^2}{\sqrt{25 - x^2}} dx.$$

Question 2 – Solution



The radical has the form $\sqrt{a^2 - x^2}$, so let

$$x = 5 \sin \theta, \quad dx = 5 \cos \theta \, d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\sqrt{25 - x^2} = \sqrt{25 - 25 \sin^2 \theta} = 5 \cos \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{25 \sin^2 \theta}{5 \cos \theta} (5 \cos \theta \, d\theta) \\ &= 25 \int \sin^2 \theta \, d\theta \\ &= \frac{25}{2} \int (1 - \cos 2\theta) \, d\theta \\ &= \frac{25}{2} \theta - \frac{25}{4} \sin 2\theta + C. \end{aligned}$$

Since $\theta = \arcsin(x/5)$ and

$$\sin 2\theta = 2 \sin \theta \cos \theta = 2 \left(\frac{x}{5} \right) \left(\frac{\sqrt{25 - x^2}}{5} \right),$$

we obtain

$$\boxed{I = \frac{25}{2} \arcsin \left(\frac{x}{5} \right) - \frac{x}{2} \sqrt{25 - x^2} + C}.$$

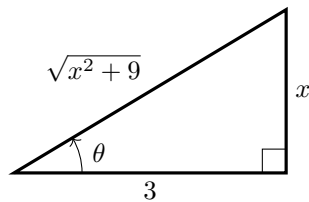
Question 3

Find the total area under

$$y = (x^2 + 9)^{-3/2}$$

on $0 \leq x < \infty$.

Question 3 – Solution



The required area is the improper integral

$$I = \lim_{b \rightarrow \infty} \int_0^b \frac{dx}{(x^2 + 9)^{3/2}}.$$

Let $x = 3 \tan \theta$. Then

$$dx = 3 \sec^2 \theta d\theta, \quad (x^2 + 9)^{3/2} = 27 \sec^3 \theta.$$

The bounds transform as $x = 0 \Rightarrow \theta = 0$ and $x \rightarrow \infty \Rightarrow \theta \rightarrow \pi/2$. Thus,

$$\begin{aligned} I &= \frac{1}{9} \int_0^{\pi/2} \cos \theta d\theta \\ &= \frac{1}{9} [\sin \theta]_0^{\pi/2} \\ &= \frac{1}{9}. \end{aligned}$$

The integrand is positive, so this convergent improper integral equals the geometric area. Therefore,

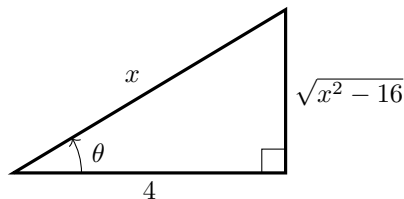
$$\boxed{I = \frac{1}{9}}.$$

Question 4

Evaluate using a trigonometric substitution:

$$\int \frac{dx}{x^2 \sqrt{x^2 - 16}}, \quad x > 4.$$

Question 4 – Solution



Because the radical has the form $\sqrt{x^2 - 16}$, let

$$x = 4 \sec \theta, \quad dx = 4 \sec \theta \tan \theta \, d\theta.$$

Since $x > 4$, take $0 < \theta < \pi/2$. Then

$$\sqrt{x^2 - 16} = \sqrt{16 \sec^2 \theta - 16} = 4 \tan \theta.$$

Therefore,

$$\begin{aligned} I &= \int \frac{4 \sec \theta \tan \theta \, d\theta}{(16 \sec^2 \theta)(4 \tan \theta)} \\ &= \frac{1}{16} \int \cos \theta \, d\theta \\ &= \frac{1}{16} \sin \theta + C. \end{aligned}$$

From the reference triangle,

$$\sec \theta = \frac{x}{4}, \quad \sin \theta = \frac{\sqrt{x^2 - 16}}{x}.$$

Hence,

$$\boxed{I = \frac{\sqrt{x^2 - 16}}{16x} + C}.$$

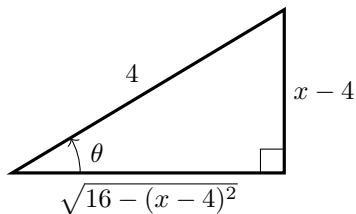
Differentiation recovers $1/(x^2\sqrt{x^2 - 16})$.

Question 5

Complete the square, choose an appropriate trigonometric substitution, and evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Question 5 – Solution



Complete the square inside the radical:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) \\ &= -((x - 4)^2 - 16) \\ &= 16 - (x - 4)^2. \end{aligned}$$

Let

$$x - 4 = 4 \sin \theta, \quad dx = 4 \cos \theta \, d\theta.$$

Choose $-\pi/2 < \theta < \pi/2$, so $\cos \theta > 0$. Then

$$\begin{aligned} \sqrt{16 - (x - 4)^2} &= \sqrt{16 - 16 \sin^2 \theta} \\ &= 4 \cos \theta. \end{aligned}$$

Thus,

$$\begin{aligned} I &= \int \frac{4 \cos \theta \, d\theta}{4 \cos \theta} \\ &= \int d\theta = \theta + C. \end{aligned}$$

Since $\sin \theta = (x - 4)/4$,

$$I = \arcsin \left(\frac{x - 4}{4} \right) + C.$$

This formula applies on intervals inside $0 < x < 8$, where the radical is real and nonzero.

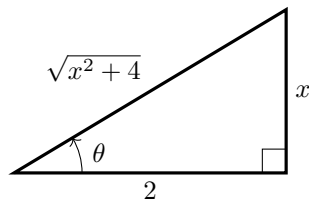
Question 6

Evaluate

$$\int \sqrt{x^2 + 4} \, dx$$

using trigonometric substitution, and verify the result by differentiation.

Question 6 – Solution



Let $x = 2 \tan \theta$, with $-\pi/2 < \theta < \pi/2$. Then

$$dx = 2 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 4} = 2 \sec \theta.$$

Therefore,

$$I = 4 \int \sec^3 \theta d\theta.$$

Using $\int \sec^3 \theta d\theta = \frac{1}{2}(\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|)$,

$$I = 2 \sec \theta \tan \theta + 2 \ln |\sec \theta + \tan \theta| + C.$$

Now $\tan \theta = x/2$ and $\sec \theta = \sqrt{x^2 + 4}/2$. Hence,

$$\begin{aligned} I &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln \left| \frac{x + \sqrt{x^2 + 4}}{2} \right| + C \\ &= \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C, \end{aligned}$$

where $-2 \ln 2$ is absorbed into C . Thus,

$$\boxed{I = \frac{x}{2} \sqrt{x^2 + 4} + 2 \ln |x + \sqrt{x^2 + 4}| + C}.$$

Differentiating and simplifying gives $I' = \sqrt{x^2 + 4}$, as required.

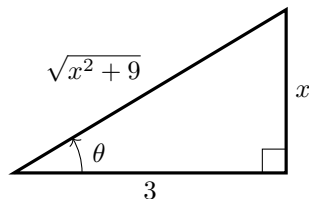
Question 7

For

$$\int \frac{dx}{\sqrt{x^2 + 9}},$$

two substitutions are proposed: $x = 3 \tan \theta$ and $x = 3 \sin \theta$. Select the efficient substitution, explain the identity it uses, and evaluate the integral.

Question 7 – Solution



Choose

$$x = 3 \tan \theta.$$

This substitution uses $1 + \tan^2 \theta = \sec^2 \theta$:

$$x^2 + 9 = 9 \tan^2 \theta + 9 = 9 \sec^2 \theta.$$

Taking $-\pi/2 < \theta < \pi/2$ gives $\sec \theta > 0$, so

$$dx = 3 \sec^2 \theta d\theta, \quad \sqrt{x^2 + 9} = 3 \sec \theta.$$

Hence,

$$\begin{aligned} I &= \int \frac{3 \sec^2 \theta}{3 \sec \theta} d\theta \\ &= \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| + C. \end{aligned}$$

Since $\tan \theta = x/3$ and $\sec \theta = \sqrt{x^2 + 9}/3$,

$$\begin{aligned} I &= \ln \left| \frac{x + \sqrt{x^2 + 9}}{3} \right| + C \\ &= \ln |x + \sqrt{x^2 + 9}| + C. \end{aligned}$$

The constant $-\ln 3$ is absorbed into C . Thus,

$$\int \frac{dx}{\sqrt{x^2 + 9}} = \ln |x + \sqrt{x^2 + 9}| + C.$$

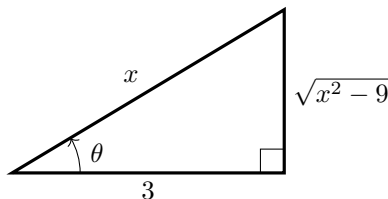
The sine substitution does not turn $x^2 + 9$ into a single squared trigonometric function, so it does not simplify the radical.

Question 8

Evaluate the definite integral exactly:

$$\int_3^5 \frac{\sqrt{x^2 - 9}}{x} dx.$$

Question 8 – Solution



Let

$$x = 3 \sec \theta, \quad dx = 3 \sec \theta \tan \theta d\theta.$$

For $x \geq 3$, take $0 \leq \theta < \pi/2$. Then

$$\sqrt{x^2 - 9} = 3 \tan \theta.$$

Transform the bounds:

$$x = 3 \Rightarrow \sec \theta = 1 \Rightarrow \theta = 0, \quad x = 5 \Rightarrow \cos \theta = \frac{3}{5} \Rightarrow \theta = \arccos\left(\frac{3}{5}\right).$$

Thus,

$$\begin{aligned} I &= \int_0^{\arccos(3/5)} \frac{3 \tan \theta}{3 \sec \theta} (3 \sec \theta \tan \theta) d\theta \\ &= 3 \int_0^{\arccos(3/5)} \tan^2 \theta d\theta \\ &= 3 [\tan \theta - \theta]_0^{\arccos(3/5)}. \end{aligned}$$

At the upper bound, a 3-4-5 triangle gives $\tan \theta = 4/3$. Therefore,

$$\begin{aligned} I &= 3 \left(\frac{4}{3} - \arccos\left(\frac{3}{5}\right) \right) \\ &= 4 - 3 \arccos\left(\frac{3}{5}\right). \end{aligned}$$

Hence,

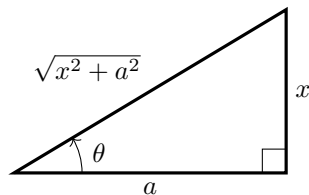
$$\boxed{I = 4 - 3 \arccos\left(\frac{3}{5}\right)}.$$

Question 9

For $a > 0$, derive the antiderivative

$$\int \frac{dx}{(a^2 + x^2)^{3/2}}.$$

Question 9 – Solution



Let

$$x = a \tan \theta, \quad dx = a \sec^2 \theta d\theta.$$

Since $a > 0$, choose $-\pi/2 < \theta < \pi/2$. Then

$$\begin{aligned} (a^2 + x^2)^{3/2} &= (a^2 + a^2 \tan^2 \theta)^{3/2} \\ &= (a^2 \sec^2 \theta)^{3/2} = a^3 \sec^3 \theta. \end{aligned}$$

Therefore,

$$\begin{aligned} I &= \int \frac{a \sec^2 \theta}{a^3 \sec^3 \theta} d\theta \\ &= \frac{1}{a^2} \int \cos \theta d\theta \\ &= \frac{1}{a^2} \sin \theta + C. \end{aligned}$$

From a reference triangle,

$$\tan \theta = \frac{x}{a}, \quad \sin \theta = \frac{x}{\sqrt{a^2 + x^2}}.$$

Thus,

$$I = \frac{x}{a^2 \sqrt{a^2 + x^2}} + C.$$

Indeed,

$$\frac{d}{dx} \left(\frac{x}{a^2 \sqrt{a^2 + x^2}} \right) = \frac{1}{(a^2 + x^2)^{3/2}},$$

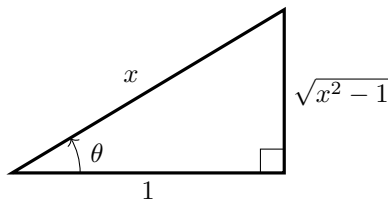
which verifies the result.

Question 10

Use a trigonometric or hyperbolic substitution to evaluate

$$\int \frac{\sqrt{x^2 - 1}}{x^2} dx, \quad x > 1.$$

Question 10 – Solution



Because $x > 1$, let

$$x = \sec \theta, \quad 0 < \theta < \frac{\pi}{2}.$$

Then

$$dx = \sec \theta \tan \theta \, d\theta, \quad \sqrt{x^2 - 1} = \tan \theta.$$

Substitution gives

$$\begin{aligned} I &= \int \frac{\tan \theta}{\sec^2 \theta} (\sec \theta \tan \theta) \, d\theta \\ &= \int \frac{\tan^2 \theta}{\sec \theta} \, d\theta \\ &= \int \frac{\sec^2 \theta - 1}{\sec \theta} \, d\theta \\ &= \int (\sec \theta - \cos \theta) \, d\theta \\ &= \ln |\sec \theta + \tan \theta| - \sin \theta + C. \end{aligned}$$

Back-substitute using

$$\sec \theta = x, \quad \tan \theta = \sqrt{x^2 - 1}, \quad \sin \theta = \frac{\sqrt{x^2 - 1}}{x}.$$

Since $x > 1$, the logarithm's argument is positive. Therefore,

$$I = \ln \left(x + \sqrt{x^2 - 1} \right) - \frac{\sqrt{x^2 - 1}}{x} + C.$$