

Question 1

Evaluate

$$\int \sin^5(x) \cos^4(x) dx.$$

Explain why separating one factor of $\sin(x)$ is an efficient strategy.

Question 1 – Solution

The power of sine is odd, so separate one sine factor:

$$\sin^5(x) = \sin^4(x) \sin(x).$$

The remaining even power can be converted using $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^4(x) = (1 - \cos^2(x))^2.$$

Thus,

$$\int \sin^5(x) \cos^4(x) dx = \int (1 - \cos^2(x))^2 \cos^4(x) \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

Then the integral becomes

$$\begin{aligned} - \int (1 - u^2)^2 u^4 du &= - \int (1 - 2u^2 + u^4) u^4 du \\ &= - \int (u^4 - 2u^6 + u^8) du \\ &= -\frac{u^5}{5} + \frac{2u^7}{7} - \frac{u^9}{9} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \sin^5(x) \cos^4(x) dx = -\frac{\cos^5(x)}{5} + \frac{2 \cos^7(x)}{7} - \frac{\cos^9(x)}{9} + C.}$$

Separating one sine factor is efficient because it supplies the differential $-\sin(x) dx$ while every remaining factor becomes a polynomial in u .

Question 2

Use power-reduction identities to evaluate

$$\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx.$$

Question 2 – Solution

Begin with the double-angle identity

$$\sin(2x) = 2 \sin(x) \cos(x).$$

Squaring both sides gives

$$\sin^2(x) \cos^2(x) = \frac{1}{4} \sin^2(2x).$$

Now use $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$ with $\theta = 2x$:

$$\sin^2(x) \cos^2(x) = \frac{1}{8} (1 - \cos(4x)).$$

Therefore,

$$\begin{aligned} \int_0^{\pi/2} \sin^2(x) \cos^2(x) dx &= \frac{1}{8} \int_0^{\pi/2} (1 - \cos(4x)) dx \\ &= \frac{1}{8} \left[x - \frac{1}{4} \sin(4x) \right]_0^{\pi/2}. \end{aligned}$$

Evaluate the endpoints:

$$\begin{aligned} \frac{1}{8} \left[\left(\frac{\pi}{2} - \frac{1}{4} \sin(2\pi) \right) - \left(0 - \frac{1}{4} \sin(0) \right) \right] &= \frac{1}{8} \left(\frac{\pi}{2} \right) \\ &= \frac{\pi}{16}. \end{aligned}$$

Hence,

$$\boxed{\int_0^{\pi/2} \sin^2(x) \cos^2(x) dx = \frac{\pi}{16}}.$$

Question 3

Find the average value of

$$f(x) = \sin^4(x)$$

over one full period.

Question 3 – Solution

We may use the interval $[0, 2\pi]$, whose length is 2π . The average-value formula is

$$f_{\text{avg}} = \frac{1}{2\pi} \int_0^{2\pi} \sin^4(x) dx.$$

Apply the power-reduction identity twice:

$$\begin{aligned} \sin^4(x) &= \left(\frac{1 - \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 - 2\cos(2x) + \cos^2(2x)) \\ &= \frac{1}{4} \left(1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3 - 4\cos(2x) + \cos(4x)}{8}. \end{aligned}$$

Thus,

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{2\pi} \int_0^{2\pi} \frac{3 - 4\cos(2x) + \cos(4x)}{8} dx \\ &= \frac{1}{16\pi} \left[3x - 2\sin(2x) + \frac{1}{4}\sin(4x) \right]_0^{2\pi} \\ &= \frac{1}{16\pi} (6\pi) = \frac{3}{8}. \end{aligned}$$

The sine terms vanish because their values are zero at both endpoints. Therefore,

$$\boxed{f_{\text{avg}} = \frac{3}{8}}.$$

Question 4

Evaluate

$$\int \tan^7(x) \sec^2(x) \, dx$$

without converting the integrand entirely to sines and cosines.

Question 4 – Solution

The factor $\sec^2(x) dx$ is exactly the differential of $\tan(x)$, so let

$$u = \tan(x), \quad du = \sec^2(x) dx.$$

Then

$$\begin{aligned} \int \tan^7(x) \sec^2(x) dx &= \int u^7 du \\ &= \frac{u^8}{8} + C. \end{aligned}$$

Substitute $u = \tan(x)$:

$$\boxed{\int \tan^7(x) \sec^2(x) dx = \frac{1}{8} \tan^8(x) + C}.$$

As a check,

$$\begin{aligned} \frac{d}{dx} \left[\frac{1}{8} \tan^8(x) \right] &= \frac{1}{8} \cdot 8 \tan^7(x) \sec^2(x) \\ &= \tan^7(x) \sec^2(x), \end{aligned}$$

which is the original integrand.

Question 5

Evaluate

$$\int \sec^5(x) \tan^3(x) \, dx$$

by saving a factor of $\sec(x) \tan(x)$ for substitution.

Question 5 – Solution

Separate one factor of $\sec(x) \tan(x)$:

$$\sec^5(x) \tan^3(x) = \sec^4(x) \tan^2(x) (\sec(x) \tan(x)).$$

Use $\tan^2(x) = \sec^2(x) - 1$ to rewrite the remaining tangent power:

$$\sec^4(x) \tan^2(x) = \sec^4(x) (\sec^2(x) - 1).$$

Now let

$$u = \sec(x), \quad du = \sec(x) \tan(x) dx.$$

The integral becomes

$$\begin{aligned} \int u^4(u^2 - 1) du &= \int (u^6 - u^4) du \\ &= \frac{u^7}{7} - \frac{u^5}{5} + C. \end{aligned}$$

Substituting $u = \sec(x)$ gives

$$\boxed{\int \sec^5(x) \tan^3(x) dx = \frac{1}{7} \sec^7(x) - \frac{1}{5} \sec^5(x) + C}.$$

Saving $\sec(x) \tan(x)$ is useful because it supplies du , while the identity converts all remaining factors into powers of u .

Question 6

Use a product-to-sum identity to evaluate

$$\int \sin(5x) \cos(2x) \, dx.$$

Question 6 – Solution

Use the product-to-sum identity

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

With $A = 5x$ and $B = 2x$,

$$\sin(5x) \cos(2x) = \frac{1}{2} [\sin(7x) + \sin(3x)].$$

Therefore,

$$\begin{aligned} \int \sin(5x) \cos(2x) dx &= \frac{1}{2} \int [\sin(7x) + \sin(3x)] dx \\ &= \frac{1}{2} \left(-\frac{1}{7} \cos(7x) - \frac{1}{3} \cos(3x) \right) + C \\ &= -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C. \end{aligned}$$

Hence,

$$\boxed{\int \sin(5x) \cos(2x) dx = -\frac{1}{14} \cos(7x) - \frac{1}{6} \cos(3x) + C}.$$

Differentiating gives

$$\frac{1}{2} \sin(7x) + \frac{1}{2} \sin(3x) = \sin(5x) \cos(2x),$$

confirming the answer by the same product-to-sum identity.

Question 7

Determine all integers $m, n \geq 1$ for which

$$\int_0^{2\pi} \sin(mx) \cos(nx) \, dx \neq 0.$$

Justify your conclusion using a product-to-sum identity.

Question 7 – Solution

Use

$$\sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)].$$

Setting $A = mx$ and $B = nx$ gives

$$\sin(mx) \cos(nx) = \frac{1}{2} [\sin((m + n)x) + \sin((m - n)x)]. \quad (1)$$

For every nonzero integer k ,

$$\begin{aligned} \int_0^{2\pi} \sin(kx) dx &= \left[-\frac{\cos(kx)}{k} \right]_0^{2\pi} \\ &= \frac{-\cos(2\pi k) + \cos(0)}{k} = 0, \end{aligned}$$

because $\cos(2\pi k) = 1$. Since $m + n \geq 2$, the first sine term in equation (1) always integrates to zero.

If $m \neq n$, then $m - n$ is a nonzero integer, so the second term also integrates to zero. If $m = n$, then

$$\sin((m - n)x) = \sin(0) = 0,$$

so it again contributes zero. Thus, in every case,

$$\int_0^{2\pi} \sin(mx) \cos(nx) dx = 0.$$

Therefore, there are no integers $m, n \geq 1$ for which the integral is nonzero:

There are no such integer pairs (m, n) .

Question 8

Evaluate

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx$$

using a substitution that produces reciprocal powers of the new variable.

Question 8 – Solution

Separate one sine factor and use $\sin^2(x) = 1 - \cos^2(x)$:

$$\sin^3(x) = \sin(x)(1 - \cos^2(x)).$$

Then

$$\int \frac{\sin^3(x)}{\cos^4(x)} dx = \int \frac{1 - \cos^2(x)}{\cos^4(x)} \sin(x) dx.$$

Let

$$u = \cos(x), \quad du = -\sin(x) dx.$$

The integral becomes

$$\begin{aligned} - \int \frac{1 - u^2}{u^4} du &= - \int (u^{-4} - u^{-2}) du \\ &= - \left(-\frac{1}{3}u^{-3} + u^{-1} \right) + C \\ &= \frac{1}{3}u^{-3} - u^{-1} + C. \end{aligned}$$

Substituting $u = \cos(x)$ gives

$$\boxed{\int \frac{\sin^3(x)}{\cos^4(x)} dx = \frac{1}{3 \cos^3(x)} - \frac{1}{\cos(x)} + C}.$$

Equivalently, the answer is $\frac{1}{3} \sec^3(x) - \sec(x) + C$ on intervals where the original integrand is defined.

Question 9

Find constants A and B such that

$$\cos^4(x) = A + B \cos(2x) + \frac{1}{8} \cos(4x).$$

Then use the identity to evaluate $\int \cos^4(x) \, dx$.

Question 9 – Solution

Begin with the power-reduction identity

$$\cos^2(x) = \frac{1 + \cos(2x)}{2}.$$

Squaring both sides,

$$\begin{aligned}\cos^4(x) &= \left(\frac{1 + \cos(2x)}{2} \right)^2 \\ &= \frac{1}{4} (1 + 2 \cos(2x) + \cos^2(2x)) .\end{aligned}$$

Apply power reduction again:

$$\cos^2(2x) = \frac{1 + \cos(4x)}{2}.$$

Therefore,

$$\begin{aligned}\cos^4(x) &= \frac{1}{4} \left(1 + 2 \cos(2x) + \frac{1 + \cos(4x)}{2} \right) \\ &= \frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x).\end{aligned}$$

Matching this with the required form gives

$$\boxed{A = \frac{3}{8}, \quad B = \frac{1}{2}}.$$

Now integrate the identity term by term:

$$\begin{aligned}\int \cos^4(x) dx &= \int \left(\frac{3}{8} + \frac{1}{2} \cos(2x) + \frac{1}{8} \cos(4x) \right) dx \\ &= \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C.\end{aligned}$$

Hence,

$$\boxed{\int \cos^4(x) dx = \frac{3x}{8} + \frac{1}{4} \sin(2x) + \frac{1}{32} \sin(4x) + C}.$$

Question 10

A signal is defined by

$$s(t) = \sin(3t) + \sin(5t).$$

Compute its energy over $0 \leq t \leq 2\pi$:

$$E = \int_0^{2\pi} s^2(t) dt.$$

Question 10 – Solution

First expand the square:

$$\begin{aligned} E &= \int_0^{2\pi} (\sin(3t) + \sin(5t))^2 dt \\ &= \int_0^{2\pi} [\sin^2(3t) + 2\sin(3t)\sin(5t) + \sin^2(5t)] dt. \end{aligned} \tag{1}$$

For the cross term, use

$$2\sin(A)\sin(B) = \cos(A - B) - \cos(A + B).$$

Thus,

$$\begin{aligned} 2 \int_0^{2\pi} \sin(3t)\sin(5t) dt &= \int_0^{2\pi} (\cos(2t) - \cos(8t)) dt \\ &= \left[\frac{1}{2}\sin(2t) - \frac{1}{8}\sin(8t) \right]_0^{2\pi} = 0. \end{aligned}$$

For each squared term, use $\sin^2(kt) = \frac{1}{2}(1 - \cos(2kt))$:

$$\begin{aligned} \int_0^{2\pi} \sin^2(kt) dt &= \frac{1}{2} \int_0^{2\pi} (1 - \cos(2kt)) dt \\ &= \frac{1}{2} \left[t - \frac{\sin(2kt)}{2k} \right]_0^{2\pi} = \pi \end{aligned}$$

for any positive integer k . In particular, the $k = 3$ and $k = 5$ terms each contribute π . Substituting into equation (1),

$$E = \pi + 0 + \pi = 2\pi.$$

Therefore,

$$\boxed{E = 2\pi}.$$