

Question 1

Given

$$f(0) = 1, f(1) = 2, f(2) = 5, f(3) = 10, f(4) = 17,$$

use T_4 to approximate $\int_0^4 f(x) dx$.

Question 1 – Solution

Step 1: Find the subinterval width.

$$h = \frac{b - a}{n} = \frac{4 - 0}{4} = 1.$$

Step 2: Write the trapezoidal formula.

$$T_4 = \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)].$$

Step 3: Substitute the table values.

$$\begin{aligned} T_4 &= \frac{1}{2} [1 + 2(2) + 2(5) + 2(10) + 17] \\ &= \frac{1}{2} [1 + 4 + 10 + 20 + 17] \\ &= \frac{1}{2} (52) = 26. \end{aligned}$$

$T_4 = 26$

Question 2

Use M_4 to approximate

$$\int_0^2 e^{-x^2} dx.$$

Give the exponential form and a decimal.

Question 2 – Solution

Step 1: Find the width.

$$h = \frac{2 - 0}{4} = 0.5.$$

Step 2: Find the four midpoints.

$$0.25, \quad 0.75, \quad 1.25, \quad 1.75.$$

Step 3: Apply the midpoint formula.

$$\begin{aligned} M_4 &= h \sum_{i=1}^4 f(m_i) \\ &= 0.5 \left(e^{-(0.25)^2} + e^{-(0.75)^2} + e^{-(1.25)^2} + e^{-(1.75)^2} \right). \end{aligned}$$

Step 4: Evaluate.

$$M_4 \approx 0.88278895.$$

$M_4 \approx 0.8828$

Question 3

Use Simpson's Rule with $n = 4$ to approximate

$$\int_0^2 \sqrt{1+x^3} \, dx.$$

Question 3 – Solution

Step 1: Find the width and nodes.

$$h = \frac{2-0}{4} = 0.5, \quad x_i = 0, 0.5, 1, 1.5, 2.$$

Step 2: Evaluate $f(x) = \sqrt{1+x^3}$.

$$\begin{array}{lll} f(0) = 1, & f(0.5) = \sqrt{1.125}, & f(1) = \sqrt{2}, \\ f(1.5) = \sqrt{4.375}, & f(2) = 3. & \end{array}$$

Step 3: Apply Simpson's Rule.

$$\begin{aligned} S_4 &= \frac{h}{3} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \\ &= \frac{0.5}{3} [1 + 4\sqrt{1.125} + 2\sqrt{2} + 4\sqrt{4.375} + 3] \\ &\approx 3.23961135. \end{aligned}$$

$$\boxed{S_4 \approx 3.2396}$$

Question 4

For increasing f , explain why

$$L_n \leq \int_a^b f(x) dx \leq R_n.$$

Then bound $\int_0^1 e^x dx$ using $n = 2$.

Question 4 – Solution

Step 1: Justify the inequality. On each subinterval, an increasing function satisfies

$$f(x_{i-1}) \leq f(x) \leq f(x_i).$$

Multiplying by $\Delta x > 0$ and adding over all subintervals gives

$$L_n \leq \int_a^b f(x) dx \leq R_n.$$

Step 2: Find the width and nodes.

$$\Delta x = \frac{1-0}{2} = \frac{1}{2}, \quad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1.$$

Step 3: Compute both sums.

$$\begin{aligned} L_2 &= \frac{1}{2}[f(0) + f(1/2)] = \frac{1}{2}(1 + e^{1/2}) \approx 1.3244, \\ R_2 &= \frac{1}{2}[f(1/2) + f(1)] = \frac{1}{2}(e^{1/2} + e) \approx 2.1835. \end{aligned}$$

Step 4: State the bound.

$$1.3244 \leq \int_0^1 e^x dx \leq 2.1835$$

Question 5

For $\int_0^2 x^4 dx$, determine whether T_n and M_n are overestimates or underestimates. Use concavity.

Question 5 – Solution

Step 1: Determine concavity. For $f(x) = x^4$,

$$f'(x) = 4x^3, \quad f''(x) = 12x^2.$$

On $[0, 2]$, $f''(x) \geq 0$, so the graph is concave up.

Step 2: Interpret the trapezoidal approximation. For a concave-up function, each secant line lies above the graph. Therefore,

$$T_n > \int_0^2 x^4 dx.$$

Step 3: Interpret the midpoint approximation. For a concave-up function, midpoint rectangles lie below the graph on average. Therefore,

$$M_n < \int_0^2 x^4 dx.$$

$T_n > I, \quad M_n < I$

Question 6

How large must n be for T_n to approximate

$$\int_0^1 e^x dx$$

within 10^{-4} ? Use $|f''(x)| \leq e$.

Question 6 – Solution

Step 1: Write the trapezoidal error bound.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Here $K = e$ and $b - a = 1$, so

$$|E_T| \leq \frac{e}{12n^2}.$$

Step 2: Require the bound to be at most 10^{-4} .

$$\frac{e}{12n^2} \leq 10^{-4},$$

$$e \leq 0.0012n^2,$$

$$n^2 \geq \frac{e}{0.0012} \approx 2265.23,$$

$$n \geq \sqrt{2265.23} \approx 47.594.$$

Step 3: Choose the smallest integer.

$$\boxed{n = 48}$$

Question 7

Show that Simpson's Rule with $n = 2$ is exact for

$$\int_0^2 x^3 dx.$$

Question 7 – Solution

Step 1: Find the width and nodes.

$$h = \frac{2-0}{2} = 1, \quad x_0 = 0, \ x_1 = 1, \ x_2 = 2.$$

Step 2: Apply Simpson's Rule.

$$\begin{aligned} S_2 &= \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] \\ &= \frac{1}{3}[0^3 + 4(1^3) + 2^3] \\ &= \frac{1}{3}(0 + 4 + 8) = 4. \end{aligned}$$

Step 3: Evaluate the exact integral.

$$\begin{aligned} I &= \int_0^2 x^3 dx = \left[\frac{x^4}{4} \right]_0^2 \\ &= \frac{16}{4} - 0 = 4. \end{aligned}$$

Since $S_2 = I$, Simpson's Rule is exact here.

$$\boxed{S_2 = I = 4}$$

Question 8

For an increasing function on $[0, 3]$, suppose

$$L_6 = 7.2, \quad R_6 = 8.7.$$

Find T_6 and bound the integral.

Question 8 – Solution

Step 1: Use the trapezoidal-average identity. With equal-width subintervals,

$$T_n = \frac{L_n + R_n}{2}.$$

Thus

$$\begin{aligned} T_6 &= \frac{7.2 + 8.7}{2} \\ &= \frac{15.9}{2} = 7.95. \end{aligned}$$

Step 2: Use monotonicity to bound the integral. Because f is increasing, left endpoints underestimate and right endpoints overestimate:

$$L_6 \leq I \leq R_6.$$

Therefore,

$$7.2 \leq I \leq 8.7.$$

$7.2 \leq I \leq 8.7, \quad T_6 = 7.95$

Question 9

A velocity is measured every second:

t (s)	0	1	2	3	4
v (m/s)	0	3	5	6	4

Use T_4 to estimate displacement.

Question 9 – Solution

Step 1: Find the time width.

$$h = \frac{4 - 0}{4} = 1 \text{ second.}$$

Step 2: Apply the trapezoidal formula.

$$\begin{aligned} T_4 &= \frac{h}{2}[v_0 + 2v_1 + 2v_2 + 2v_3 + v_4] \\ &= \frac{1}{2}[0 + 2(3) + 2(5) + 2(6) + 4] \\ &= \frac{1}{2}[0 + 6 + 10 + 12 + 4] \\ &= \frac{1}{2}(32) = 16. \end{aligned}$$

Step 3: Include units. Velocity times time gives displacement in meters.

$$\boxed{16 \text{ m}}$$

Question 10

For Simpson's Rule on $[0, \pi]$, suppose

$$|f^{(4)}(x)| \leq 1.$$

Find the smallest even n that guarantees error at most 10^{-6} .

Question 10 – Solution

Step 1: Write Simpson's error bound.

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$

Here $K = 1$ and $b - a = \pi$, so

$$|E_S| \leq \frac{\pi^5}{180n^4}.$$

Step 2: Require the bound to be at most 10^{-6} .

$$\begin{aligned}\frac{\pi^5}{180n^4} &\leq 10^{-6}, \\ n^4 &\geq \frac{\pi^5}{180 \cdot 10^{-6}}, \\ n &\geq \left(\frac{\pi^5}{180 \cdot 10^{-6}} \right)^{1/4} \\ &\approx 36.109.\end{aligned}$$

Step 3: Enforce Simpson's even- n requirement. The first even integer at least 36.109 is 38.

$$\boxed{n = 38}$$