

Question 1

Evaluate by tabular integration:

$$\int x^3 e^{2x} dx.$$

Question 1 – Solution

We use tabular integration because repeatedly differentiating the polynomial x^3 eventually gives zero. Differentiate in the D column and integrate in the I column. The signs alternate, beginning with a plus sign.

Sign	D	I
+	x^3	$\frac{e^{2x}}{2}$
–	$3x^2$	$\frac{e^{2x}}{4}$
+	$6x$	$\frac{e^{2x}}{8}$
–	6	$\frac{e^{2x}}{16}$
	0	

For example, each entry in the integration column is obtained using

$$\int e^{2x} dx = \frac{1}{2}e^{2x},$$

so every step down that column introduces another factor of $\frac{1}{2}$.

Now multiply across each row, pairing each entry in the derivative column with the entry in the integration column on the same row, and use the indicated signs:

$$\begin{aligned} \int x^3 e^{2x} dx &= x^3 \left(\frac{e^{2x}}{2} \right) - 3x^2 \left(\frac{e^{2x}}{4} \right) + 6x \left(\frac{e^{2x}}{8} \right) - 6 \left(\frac{e^{2x}}{16} \right) + C \\ &= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{6x e^{2x}}{8} - \frac{6e^{2x}}{16} + C \\ &= e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C. \end{aligned}$$

Therefore,

$$\boxed{\int x^3 e^{2x} dx = e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) + C}$$

As a check, differentiating the answer gives

$$\frac{d}{dx} \left[e^{2x} \left(\frac{x^3}{2} - \frac{3x^2}{4} + \frac{3x}{4} - \frac{3}{8} \right) \right] = x^3 e^{2x},$$

which is the original integrand.

Question 2

Find the unique constant a such that

$$\int_0^1 x e^{ax} dx = 1.$$

Give the defining equation and a numerical value.

Question 2 – Solution

First consider the special case $a = 0$. In that case,

$$\int_0^1 x e^{0x} dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2} \neq 1.$$

Therefore, $a = 0$ is not a solution, so we may assume that $a \neq 0$.

To evaluate the integral, use integration by parts with

$$u = x, \quad dv = e^{ax} dx.$$

Then

$$du = dx, \quad v = \frac{e^{ax}}{a}.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_0^1 x e^{ax} dx &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{1}{a} \int_0^1 e^{ax} dx \\ &= \frac{x e^{ax}}{a} \Big|_0^1 - \frac{e^{ax}}{a^2} \Big|_0^1 \\ &= \frac{e^a}{a} - \frac{e^a - 1}{a^2} \\ &= \frac{a e^a - e^a + 1}{a^2} \\ &= \frac{e^a(a - 1) + 1}{a^2}. \end{aligned}$$

We want this integral to equal 1, so

$$\begin{aligned} \frac{e^a(a - 1) + 1}{a^2} &= 1, \\ e^a(a - 1) + 1 &= a^2. \end{aligned}$$

Thus, the defining equation is

$$\boxed{e^a(a - 1) + 1 = a^2}.$$

Substituting $a = 1$ verifies that it satisfies the equation:

$$e^1(1 - 1) + 1 = 1 = 1^2.$$

Hence, $a = 1$ is a solution.

To see that it is the only solution, define

$$F(a) = \int_0^1 x e^{ax} dx.$$

Differentiating with respect to a gives

$$F'(a) = \int_0^1 x^2 e^{ax} dx > 0$$

for every real a . Therefore, F is strictly increasing and can equal 1 for at most one value of a . Since $F(1) = 1$, the unique constant is

$$\boxed{a = 1.0000}.$$

Question 3

Consider the integral

$$\int x^2 e^x dx.$$

A student makes the integration-by-parts choice

$$u = e^x, \quad dv = x^2 dx.$$

Explain why this choice does not simplify the integral. Then evaluate the integral efficiently using a better choice of u and dv .

Question 3 – Solution

Recall the integration-by-parts formula

$$\int u \, dv = uv - \int v \, du.$$

With the student's choice $u = e^x$ and $dv = x^2 \, dx$, we have

$$du = e^x \, dx, \quad v = \frac{x^3}{3}.$$

Therefore,

$$\begin{aligned} \int x^2 e^x \, dx &= e^x \left(\frac{x^3}{3} \right) - \int \frac{x^3}{3} e^x \, dx \\ &= \frac{x^3 e^x}{3} - \frac{1}{3} \int x^3 e^x \, dx. \end{aligned}$$

The polynomial degree has increased from 2 to 3, so the remaining integral is more complicated. Repeating this choice would keep increasing the degree.

Instead, choose

$$u = x^2, \quad dv = e^x \, dx, \quad du = 2x \, dx, \quad v = e^x.$$

Applying integration by parts gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - \int 2x e^x \, dx \\ &= x^2 e^x - 2 \int x e^x \, dx. \end{aligned}$$

For the remaining integral, use integration by parts again:

$$u = x, \quad dv = e^x \, dx, \quad du = dx, \quad v = e^x.$$

Thus,

$$\begin{aligned} \int x e^x \, dx &= x e^x - \int e^x \, dx \\ &= x e^x - e^x. \end{aligned}$$

Substituting this result into the original calculation gives

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2(x e^x - e^x) + C \\ &= x^2 e^x - 2x e^x + 2e^x + C \\ &= e^x (x^2 - 2x + 2) + C. \end{aligned}$$

$$\boxed{\int x^2 e^x \, dx = e^x (x^2 - 2x + 2) + C}$$

Check by differentiating:

$$\begin{aligned} \frac{d}{dx} [e^x (x^2 - 2x + 2)] &= e^x (x^2 - 2x + 2) + e^x (2x - 2) \\ &= x^2 e^x, \end{aligned}$$

which is the original integrand.

Question 4

Define

$$I_n = \int x^n e^x dx, \quad n \geq 1.$$

Use integration by parts to derive a recurrence relation connecting I_n and I_{n-1} . Then use the recurrence to evaluate I_4 .

Question 4 – Solution

Start with

$$I_n = \int x^n e^x dx.$$

To reduce the power of x , choose

$$u = x^n, \quad dv = e^x dx, \quad du = nx^{n-1} dx, \quad v = e^x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} I_n &= x^n e^x - \int e^x (nx^{n-1}) dx \\ &= x^n e^x - n \int x^{n-1} e^x dx. \end{aligned}$$

Since $I_{n-1} = \int x^{n-1} e^x dx$, the recurrence relation is

$$\boxed{I_n = x^n e^x - n I_{n-1}}.$$

Here, as usual for indefinite integrals, the constant of integration is added after the recurrence has been applied. The base case is

$$I_0 = \int e^x dx = e^x.$$

Now apply the recurrence repeatedly:

$$\begin{aligned} I_1 &= x e^x - I_0 = x e^x - e^x = e^x(x - 1), \\ I_2 &= x^2 e^x - 2I_1 \\ &= x^2 e^x - 2e^x(x - 1) = e^x(x^2 - 2x + 2), \\ I_3 &= x^3 e^x - 3I_2 \\ &= x^3 e^x - 3e^x(x^2 - 2x + 2) \\ &= e^x(x^3 - 3x^2 + 6x - 6), \\ I_4 &= x^4 e^x - 4I_3 \\ &= x^4 e^x - 4e^x(x^3 - 3x^2 + 6x - 6) \\ &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24). \end{aligned}$$

Therefore,

$$\boxed{I_4 = \int x^4 e^x dx = e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) + C}$$

Indeed, differentiating the result gives

$$\begin{aligned} \frac{d}{dx} [e^x(x^4 - 4x^3 + 12x^2 - 24x + 24)] &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24) \\ &\quad + e^x(4x^3 - 12x^2 + 24x - 24) \\ &= x^4 e^x, \end{aligned}$$

which confirms the answer.

Question 5

Evaluate the cyclic integral

$$\int e^{2x} \cos(3x) dx$$

using integration by parts. Do not guess the form of the antiderivative.

Question 5 – Solution

Let

$$I = \int e^{2x} \cos(3x) dx.$$

For the first integration by parts, choose

$$u = \cos(3x), \quad dv = e^{2x} dx, \quad du = -3 \sin(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Using $\int u dv = uv - \int v du$, we get

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) - \int \frac{1}{2} e^{2x} (-3 \sin(3x)) dx \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \int e^{2x} \sin(3x) dx. \end{aligned} \tag{1}$$

Now evaluate the remaining integral. Let

$$J = \int e^{2x} \sin(3x) dx.$$

Use integration by parts again with

$$u = \sin(3x), \quad dv = e^{2x} dx, \quad du = 3 \cos(3x) dx, \quad v = \frac{1}{2} e^{2x}.$$

Then

$$\begin{aligned} J &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} \int e^{2x} \cos(3x) dx \\ &= \frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I. \end{aligned} \tag{2}$$

Substitute equation (2) into equation (1):

$$\begin{aligned} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{2} \left(\frac{1}{2} e^{2x} \sin(3x) - \frac{3}{2} I \right) \\ &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x) - \frac{9}{4} I. \end{aligned}$$

Move the I -term to the left and solve:

$$\begin{aligned} I + \frac{9}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ \frac{13}{4} I &= \frac{1}{2} e^{2x} \cos(3x) + \frac{3}{4} e^{2x} \sin(3x), \\ I &= \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)). \end{aligned}$$

Therefore,

$$\boxed{\int e^{2x} \cos(3x) dx = \frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) + C}$$

To check, differentiate the antiderivative:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^{2x}}{13} (2 \cos(3x) + 3 \sin(3x)) \right] &= \frac{e^{2x}}{13} [2(2 \cos(3x) + 3 \sin(3x)) \\ &\quad - 6 \sin(3x) + 9 \cos(3x)] \\ &= e^{2x} \cos(3x), \end{aligned}$$

which is the original integrand.

Question 6

Use integration by parts to prove that

$$\int_0^1 \ln(x) \, dx = -1.$$

Because $\ln(x)$ is unbounded as $x \rightarrow 0^+$, be sure to treat the lower endpoint correctly as an improper integral.

Question 6 – Solution

The function $\ln(x)$ is not defined at $x = 0$ and satisfies

$$\lim_{x \rightarrow 0^+} \ln(x) = -\infty.$$

Therefore, we must interpret the integral as

$$\int_0^1 \ln(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 \ln(x) dx.$$

For $\varepsilon > 0$, evaluate the proper integral by integration by parts. Choose

$$u = \ln(x), \quad dv = dx, \quad du = \frac{1}{x} dx, \quad v = x.$$

Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 x \left(\frac{1}{x} \right) dx \\ &= x \ln(x) \Big|_{\varepsilon}^1 - \int_{\varepsilon}^1 1 dx \\ &= [x \ln(x) - x]_{\varepsilon}^1. \end{aligned}$$

Now substitute the bounds:

$$\begin{aligned} \int_{\varepsilon}^1 \ln(x) dx &= (1 \ln(1) - 1) - (\varepsilon \ln(\varepsilon) - \varepsilon) \\ &= -1 - \varepsilon \ln(\varepsilon) + \varepsilon. \end{aligned}$$

It remains to evaluate the indeterminate endpoint term. Rewrite it as

$$\varepsilon \ln(\varepsilon) = \frac{\ln(\varepsilon)}{1/\varepsilon}.$$

As $\varepsilon \rightarrow 0^+$, this has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \varepsilon \ln(\varepsilon) &= \lim_{\varepsilon \rightarrow 0^+} \frac{1/\varepsilon}{-1/\varepsilon^2} \\ &= \lim_{\varepsilon \rightarrow 0^+} (-\varepsilon) = 0. \end{aligned}$$

Also, $\lim_{\varepsilon \rightarrow 0^+} \varepsilon = 0$. Hence,

$$\begin{aligned} \int_0^1 \ln(x) dx &= \lim_{\varepsilon \rightarrow 0^+} (-1 - \varepsilon \ln(\varepsilon) + \varepsilon) \\ &= -1. \end{aligned}$$

Therefore, the improper integral converges and

$$\boxed{\int_0^1 \ln(x) dx = -1}.$$

This sign is reasonable because $\ln(x) < 0$ for $0 < x < 1$.

Question 7

Find an antiderivative F of

$$f(x) = \arcsin(x), \quad -1 < x < 1,$$

that satisfies the initial condition

$$F(0) = 2.$$

Question 7 – Solution

We begin by evaluating

$$\int \arcsin(x) \, dx.$$

Since the integrand can be viewed as $\arcsin(x) \cdot 1$, use integration by parts with

$$u = \arcsin(x), \quad dv = dx, \quad du = \frac{1}{\sqrt{1-x^2}} dx, \quad v = x.$$

Using $\int u \, dv = uv - \int v \, du$, we obtain

$$\int \arcsin(x) \, dx = x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} \, dx. \quad (1)$$

To evaluate the remaining integral, let

$$w = 1 - x^2.$$

Then

$$dw = -2x \, dx, \quad x \, dx = -\frac{1}{2} dw.$$

Therefore,

$$\begin{aligned} \int \frac{x}{\sqrt{1-x^2}} \, dx &= -\frac{1}{2} \int w^{-1/2} \, dw \\ &= -\frac{1}{2} (2w^{1/2}) \\ &= -\sqrt{w} \\ &= -\sqrt{1-x^2}. \end{aligned}$$

Substitute this result into equation (1):

$$\begin{aligned} \int \arcsin(x) \, dx &= x \arcsin(x) - (-\sqrt{1-x^2}) + C \\ &= x \arcsin(x) + \sqrt{1-x^2} + C. \end{aligned}$$

Thus, the general antiderivative is

$$F(x) = x \arcsin(x) + \sqrt{1-x^2} + C.$$

Now apply the condition $F(0) = 2$:

$$\begin{aligned} F(0) &= 0 \cdot \arcsin(0) + \sqrt{1-0^2} + C \\ &= 1 + C. \end{aligned}$$

Hence,

$$1 + C = 2 \quad \implies \quad C = 1.$$

Therefore, the required antiderivative is

$$\boxed{F(x) = x \arcsin(x) + \sqrt{1-x^2} + 1}.$$

To verify, differentiate:

$$\begin{aligned} F'(x) &= \arcsin(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} \\ &= \arcsin(x). \end{aligned}$$

Also, $F(0) = 0 + 1 + 1 = 2$, so both requirements are satisfied.

Question 8

Reverse engineer an antiderivative by finding a quadratic polynomial $P(x)$ such that

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1).$$

Determine $P(x)$ and verify that it satisfies the equation.

Question 8 – Solution

We want to choose $P(x)$ so that differentiating $e^x P(x)$ produces the given expression. By the product rule,

$$\begin{aligned}\frac{d}{dx} [e^x P(x)] &= e^x P(x) + e^x P'(x) \\ &= e^x (P(x) + P'(x)).\end{aligned}$$

Thus, the required identity

$$\frac{d}{dx} [e^x P(x)] = e^x (x^2 + 1)$$

will hold if

$$P(x) + P'(x) = x^2 + 1. \tag{1}$$

Because P is quadratic, write

$$P(x) = ax^2 + bx + c,$$

where a , b , and c are constants. Then

$$P'(x) = 2ax + b.$$

Adding $P(x)$ and $P'(x)$ gives

$$\begin{aligned}P(x) + P'(x) &= (ax^2 + bx + c) + (2ax + b) \\ &= ax^2 + (b + 2a)x + (c + b).\end{aligned}$$

Equation (1) requires this polynomial to equal

$$x^2 + 1 = 1x^2 + 0x + 1.$$

Matching coefficients of like powers of x yields

$$\begin{array}{rcl}a & = & 1, \quad \text{coefficient of } x^2, \\ b + 2a & = & 0, \quad \text{coefficient of } x, \\ c + b & = & 1, \quad \text{constant term.}\end{array}$$

Solve the equations in order. Since $a = 1$,

$$b + 2(1) = 0 \quad \implies \quad b = -2.$$

Then

$$c + (-2) = 1 \quad \implies \quad c = 3.$$

Therefore,

$$\boxed{P(x) = x^2 - 2x + 3}.$$

Finally, verify the result. Since $P'(x) = 2x - 2$,

$$\begin{aligned}\frac{d}{dx} [e^x (x^2 - 2x + 3)] &= e^x (x^2 - 2x + 3) + e^x (2x - 2) \\ &= e^x (x^2 - 2x + 3 + 2x - 2) \\ &= e^x (x^2 + 1),\end{aligned}$$

which is the required expression.

Question 9

Evaluate

$$\int_{-1}^1 x \arctan(x) dx.$$

Before using integration by parts, exploit the symmetry of the integrand to simplify the calculation.

Question 9 – Solution

Let

$$f(x) = x \arctan(x).$$

Both x and $\arctan(x)$ are odd functions. Therefore, their product is even:

$$\begin{aligned} f(-x) &= (-x) \arctan(-x) \\ &= (-x)(-\arctan(x)) \\ &= x \arctan(x) = f(x). \end{aligned}$$

Consequently,

$$\int_{-1}^1 x \arctan(x) dx = 2 \int_0^1 x \arctan(x) dx. \quad (1)$$

Now use integration by parts on the integral over $[0, 1]$. Choose

$$u = \arctan(x), \quad dv = x dx, \quad du = \frac{1}{1+x^2} dx, \quad v = \frac{x^2}{2}.$$

Then

$$\int_0^1 x \arctan(x) dx = \frac{x^2}{2} \arctan(x) \Big|_0^1 - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx.$$

Multiplying by 2 as in equation (1) gives

$$\int_{-1}^1 x \arctan(x) dx = x^2 \arctan(x) \Big|_0^1 - \int_0^1 \frac{x^2}{1+x^2} dx. \quad (2)$$

For the remaining integral, rewrite the integrand:

$$\frac{x^2}{1+x^2} = \frac{(1+x^2) - 1}{1+x^2} = 1 - \frac{1}{1+x^2}.$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{x^2}{1+x^2} dx &= \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \\ &= [x - \arctan(x)]_0^1 \\ &= 1 - \frac{\pi}{4}. \end{aligned}$$

The boundary term in equation (2) is

$$x^2 \arctan(x) \Big|_0^1 = 1^2 \arctan(1) - 0^2 \arctan(0) = \frac{\pi}{4}.$$

Substituting both results into equation (2),

$$\begin{aligned} \int_{-1}^1 x \arctan(x) dx &= \frac{\pi}{4} - \left(1 - \frac{\pi}{4} \right) \\ &= \frac{\pi}{2} - 1. \end{aligned}$$

Thus,

$$\boxed{\int_{-1}^1 x \arctan(x) dx = \frac{\pi}{2} - 1}.$$

The answer is positive, as expected, because the even integrand is nonnegative on $[-1, 1]$.

Question 10

For $n > 0$, evaluate the improper integral

$$J_n = \int_0^1 x^n \ln(x) \, dx.$$

Then describe the sign and limiting behavior of J_n as $n \rightarrow \infty$.

Question 10 – Solution

Because $\ln(x) \rightarrow -\infty$ as $x \rightarrow 0^+$, write the integral as

$$J_n = \lim_{\varepsilon \rightarrow 0^+} \int_{\varepsilon}^1 x^n \ln(x) dx.$$

For $\varepsilon > 0$, use integration by parts with

$$u = \ln(x), \quad dv = x^n dx, \quad du = \frac{1}{x} dx, \quad v = \frac{x^{n+1}}{n+1}.$$

This choice is valid because $n > 0$, so in particular $n+1 \neq 0$. Using $\int u dv = uv - \int v du$, we obtain

$$\begin{aligned} \int_{\varepsilon}^1 x^n \ln(x) dx &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{1}{n+1} \int_{\varepsilon}^1 x^n dx \\ &= \left. \frac{x^{n+1} \ln(x)}{n+1} \right|_{\varepsilon}^1 - \frac{x^{n+1}}{(n+1)^2} \Big|_{\varepsilon}^1. \end{aligned} \tag{1}$$

We must evaluate the boundary term at $x = 0$. Since $n+1 > 0$,

$$\lim_{x \rightarrow 0^+} x^{n+1} \ln(x) = \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}}.$$

This has the form $-\infty/\infty$. By l'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{\ln(x)}{x^{-(n+1)}} &= \lim_{x \rightarrow 0^+} \frac{1/x}{-(n+1)x^{-(n+2)}} \\ &= \lim_{x \rightarrow 0^+} -\frac{x^{n+1}}{n+1} = 0. \end{aligned}$$

Also, $\lim_{x \rightarrow 0^+} x^{n+1} = 0$. At the upper endpoint, $\ln(1) = 0$. Therefore, taking $\varepsilon \rightarrow 0^+$ in equation (1) gives

$$\begin{aligned} J_n &= \frac{1^{n+1} \ln(1)}{n+1} - 0 - \frac{1^{n+1} - 0}{(n+1)^2} \\ &= -\frac{1}{(n+1)^2}. \end{aligned}$$

Hence,

$$\boxed{J_n = -\frac{1}{(n+1)^2}}.$$

For every $n > 0$, $J_n < 0$, which agrees with $\ln(x) < 0$ on $0 < x < 1$. Moreover, as $n \rightarrow \infty$,

$$\lim_{n \rightarrow \infty} J_n = \lim_{n \rightarrow \infty} -\frac{1}{(n+1)^2} = 0.$$

The values increase toward zero through negative values. In short,

$$\boxed{J_n \rightarrow 0^- \quad \text{as} \quad n \rightarrow \infty}.$$