

Question 8

Suppose $\vec{q}(u) = \vec{r}(\phi(u))$ traces the same regular curve, where ϕ is continuously differentiable and strictly monotone. Prove that the arc length is unchanged by this reparameterization, including the case in which orientation is reversed.

Question 8 – Solution

Strategy Apply the chain rule to speed, then use substitution while carefully retaining the absolute value.

same trace, opposite orientations



Transformed speed

$$\vec{q}'(u) = \vec{r}'(\phi(u))\phi'(u), \quad \|\vec{q}'(u)\| = \|\vec{r}'(\phi(u))\| |\phi'(u)|.$$

Increasing case If $\phi' \geq 0$ and $\phi(\alpha) = a, \phi(\beta) = b$, substitution gives

$$\int_{\alpha}^{\beta} \|\vec{q}'(u)\| du = \int_a^b \|\vec{r}'(t)\| dt.$$

Decreasing case If $\phi' \leq 0$, then $\phi(\alpha) = b$ and $\phi(\beta) = a$. Since $|\phi'| = -\phi'$, reversing the substituted limits again produces $\int_a^b \|\vec{r}'(t)\| dt$. Thus length depends on the traced curve, not its orientation or timing.