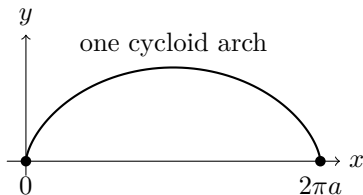


Question 6

One arch of a cycloid is parameterized by $\vec{r}(t) = \langle a(t - \sin t), a(1 - \cos t) \rangle$, $0 \leq t \leq 2\pi$, where $a > 0$. Prove that its length is $8a$. Explain the absolute-value issue in simplifying the speed.

Question 6 – Solution

Strategy Use half-angle identities and determine the sign of $\sin(t/2)$ on the interval.



Speed

$$\vec{r}' = a \langle 1 - \cos t, \sin t \rangle, \quad \|\vec{r}'\| = a\sqrt{2 - 2\cos t} = 2a \left| \sin \frac{t}{2} \right|.$$

For $0 \leq t \leq 2\pi$, $\sin(t/2) \geq 0$, so the absolute value may be removed.

Length

$$L = \int_0^{2\pi} 2a \sin \frac{t}{2} dt = \left[-4a \cos \frac{t}{2} \right]_0^{2\pi} = \boxed{8a}.$$

Endpoint note The velocity vanishes at the two cusps, but the speed is continuous and integrable, so the arch still has a well-defined finite length.