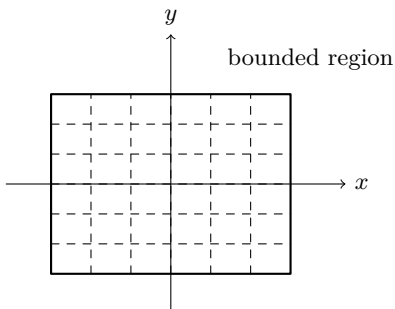


Question 10

Let $f(x, y) = \frac{xy}{1 + x^2 + y^2}$ on the closed rectangle $R = [-2, 2] \times [-3, 3]$. Prove that $|f| < 1/2$ everywhere, and explain why f must attain a maximum and minimum on R even without finding them.

Question 10 – Solution

Strategy Compare $2|xy|$ with $x^2 + y^2$, then use continuity on a closed bounded domain.



Bound From $(|x| - |y|)^2 \geq 0$, $2|xy| \leq x^2 + y^2$. Hence

$$|f(x, y)| \leq \frac{x^2 + y^2}{2(1 + x^2 + y^2)} < \boxed{\frac{1}{2}}.$$

The strict inequality comes from the added 1 in the denominator.

Existence of extrema The denominator never vanishes, so f is continuous. The rectangle R is closed and bounded (compact). By the Extreme Value Theorem, f attains both an absolute maximum and absolute minimum on R .