

Question 9

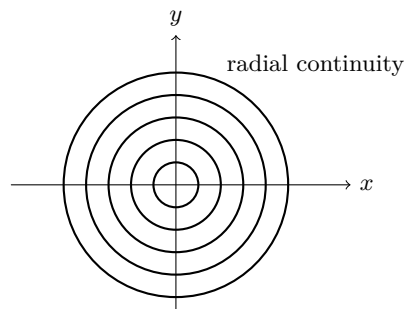
Define

$$f(x, y) = \begin{cases} \frac{x^3 - xy^2}{x^2 + y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove directly that f is continuous at the origin, despite the direction-dependent factor in its polar form.

Question 9 – Solution

Strategy Bound the absolute value by a constant times the distance to the origin.



Estimate For $(x, y) \neq (0, 0)$,

$$|f(x, y)| = |x| \frac{|x^2 - y^2|}{x^2 + y^2} \leq |x| \leq \sqrt{x^2 + y^2}.$$

Therefore $|f(x, y) - 0| \leq r$, where r is the distance to the origin.

Epsilon argument Given $\varepsilon > 0$, choose $\delta = \varepsilon$. If $0 < r < \delta$, then $|f(x, y) - f(0, 0)| < \varepsilon$. Hence $\lim f = 0 = f(0, 0)$ and f is continuous.