

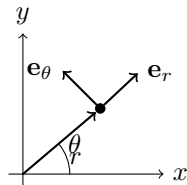
Question 8

A moving point has cylindrical coordinates $r(t), \theta(t), z(t)$. Derive its rectangular velocity and prove that its speed satisfies

$$\|\vec{v}\|^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

Question 8 – Solution

Strategy Differentiate $x = r \cos \theta$ and $y = r \sin \theta$, then simplify the sum of squares.



Velocity

$$\dot{x} = \dot{r} \cos \theta - r \dot{\theta} \sin \theta, \quad \dot{y} = \dot{r} \sin \theta + r \dot{\theta} \cos \theta, \quad \dot{z} = \dot{z}.$$

Speed identity Expanding $\dot{x}^2 + \dot{y}^2$ makes the cross terms cancel, while $\sin^2 \theta + \cos^2 \theta = 1$. Therefore

$$\|\vec{v}\|^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2 \dot{\theta}^2 + \dot{z}^2.$$

The radial, angular, and vertical velocity directions are mutually perpendicular.