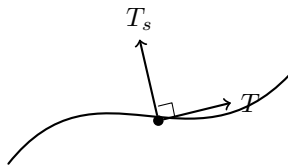


Question 5

Let $\vec{r}(s)$ be a twice-differentiable unit-speed curve. Prove that $\kappa(s) = \|\vec{r}''(s)\|$ and that the acceleration $\vec{r}''(s)$ is perpendicular to the tangent.

Question 5 – Solution

Strategy Differentiate the unit-speed identity $\vec{r}' \cdot \vec{r}' = 1$.



Orthogonality Because $\|\vec{r}'\| = 1$,

$$\vec{r}' \cdot \vec{r}' = 1 \implies 2\vec{r}' \cdot \vec{r}'' = 0.$$

Thus $\vec{r}'' \perp \vec{r}' = \mathbf{T}$.

Curvature For an arc-length parameter, $\mathbf{T} = d\vec{r}/ds$. By definition,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{d^2\vec{r}}{ds^2} \right\|.$$

Interpretation Unit speed removes tangential acceleration. Any remaining second derivative measures only the rate at which the direction turns.