

Question 4

Consider $\sum_{n=1}^{\infty} \frac{\sin n}{n}$.

- (a) Prove ordinary convergence using Dirichlet's Test.
- (b) Prove the absolute-value series diverges.
- (c) Classify the series precisely.

Question 4 – Solution

Step 1: Bound the numerator partial sums.

From the finite geometric sum of e^{in} ,

$$\left| \sum_{n=1}^N e^{in} \right| \leq \frac{2}{|1 - e^i|}.$$

Taking imaginary parts shows that $\sum_{n=1}^N \sin n$ is uniformly bounded.

Step 2: Apply Dirichlet's Test.

The sequence $1/n$ decreases to zero. Hence $\sum (\sin n)/n$ converges.

Step 3: Rule out absolute convergence.

Since $0 \leq |\sin n| \leq 1$, we have $|\sin n| \geq \sin^2 n$. Section 4-4 established that $\sum \sin^2 n/n$ diverges; therefore

$$\sum \frac{|\sin n|}{n} \geq \sum \frac{\sin^2 n}{n} = \infty.$$

Thus the series is conditionally convergent.