

Question 10

Consider $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

- (a) Show why limit comparison with $1/n$ and $1/n^p$ does not immediately decide the series.
- (b) Select a natural logarithmic benchmark or use the Integral Test.
- (c) Classify the series and state an explicit tail bound.

Question 10 – Solution

Step 1: Audit ordinary power benchmarks.

Relative to $1/n$,

$$\frac{1/[n(\ln n)^2]}{1/n} = \frac{1}{(\ln n)^2} \rightarrow 0,$$

which does not transfer the harmonic series' divergence. For any $p > 1$, the ratio to $1/n^p$ is $n^{p-1}/(\ln n)^2 \rightarrow \infty$, which does not transfer that p -series' convergence.

Step 2: Use the natural logarithmic test.

Let $f(x) = 1/[x(\ln x)^2]$. It is positive, continuous, and decreasing for $x \geq 2$. With $u = \ln x$,

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Thus the series converges by the Integral Test.

Step 3: State the tail estimate.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

The correct benchmark retains the logarithm rather than discarding it.