

Question 9

Consider $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$.

- (a) Prove a usable power bound for $\ln n$.
- (b) Compare the summand with a convergent p -series.
- (c) Explain why limit comparison with $1/n^3$ is inconclusive.

Question 9 – Solution

Step 1: Bound the logarithm.

We claim $\ln x \leq \sqrt{x}$ for $x \geq 1$. Let $h(x) = \sqrt{x} - \ln x$. Then

$$h'(x) = \frac{\sqrt{x} - 2}{2x},$$

so the minimum occurs at $x = 4$, where $h(4) = 2 - \ln 4 > 0$. The claim follows.

Step 2: Compare directly.

For $n \geq 1$,

$$0 \leq \frac{\ln n}{n^3} \leq \frac{\sqrt{n}}{n^3} = \frac{1}{n^{5/2}}.$$

The benchmark is a convergent p -series with $p = 5/2 > 1$, so Direct Comparison proves convergence.

Step 3: Explain benchmark choice.

Comparison with $1/n^3$ by ratios gives $(\ln n/n^3)/(1/n^3) = \ln n \rightarrow \infty$, outside the standard finite-positive Limit Comparison case. A slightly larger convergent power benchmark is more useful.