

Question 8

Consider $\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$.

- (a) Prove a direct lower bound using $\sqrt{n} \leq n$.
- (b) Use the correct comparison direction to classify the series.
- (c) Confirm the conclusion by limit comparison with $1/n$.

Question 8 – Solution

Step 1: Bound the denominator.

For $n \geq 1$, $\sqrt{n} \leq n$, so

$$n + \sqrt{n} \leq 2n.$$

Because both denominators are positive, taking reciprocals reverses the inequality:

$$\frac{1}{n + \sqrt{n}} \geq \frac{1}{2n}.$$

Step 2: Apply Direct Comparison.

The smaller series $\sum 1/(2n)$ diverges, so the larger given positive-term series diverges.

Step 3: Confirm by limit comparison.

$$\lim_{n \rightarrow \infty} \frac{1/(n + \sqrt{n})}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{1 + 1/\sqrt{n}} = 1.$$

This finite positive limit confirms harmonic tail behavior.