

Question 6

Consider $\sum_{n=1}^{\infty} \frac{1}{n^2 - n + 1}$.

- (a) Prove useful upper and lower quadratic bounds for the denominator.
- (b) Convert them into bounds for the summand, noting reciprocal inequality directions.
- (c) Classify the series and confirm by limit comparison.

Question 6 – Solution

Step 1: Bound the denominator.

Clearly $n^2 - n + 1 \leq n^2$. Also

$$n^2 - n + 1 \geq \frac{n^2}{2} \iff n^2 - 2n + 2 = (n - 1)^2 + 1 \geq 0.$$

Thus $n^2/2 \leq n^2 - n + 1 \leq n^2$.

Step 2: Take reciprocals carefully.

All expressions are positive, so reciprocals reverse the inequalities:

$$\frac{1}{n^2} \leq \frac{1}{n^2 - n + 1} \leq \frac{2}{n^2}.$$

The upper bound is the decisive one for convergence. Since $\sum 2/n^2$ converges, Direct Comparison proves convergence.

Step 3: Confirm asymptotically.

$$\lim_{n \rightarrow \infty} \frac{1/(n^2 - n + 1)}{1/n^2} = \lim_{n \rightarrow \infty} \frac{1}{1 - 1/n + 1/n^2} = 1.$$