

**Question 4**

Consider  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 1}}$ .

- (a) Select a benchmark using dominant powers.
- (b) Perform limit comparison.
- (c) Prove divergence again using a direct lower bound.

### Question 4 – Solution

**Step 1: Select the harmonic benchmark.**

Since  $\sqrt{n^2 + 1}$  behaves like  $n$ , choose  $b_n = 1/n$ .

**Step 2: Compute the ratio.**

$$\lim_{n \rightarrow \infty} \frac{1/\sqrt{n^2 + 1}}{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + 1/n^2}} = 1.$$

The finite positive ratio and divergence of  $\sum 1/n$  imply divergence.

**Step 3: Give a direct lower bound.**

Since  $n^2 + 1 \leq 2n^2$  for  $n \geq 1$ ,

$$\frac{1}{\sqrt{n^2 + 1}} \geq \frac{1}{\sqrt{2}n}.$$

The smaller comparison series diverges, independently confirming the verdict.