

Question 2

Consider $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+1}$.

- (a) Use dominant powers to select a harmonic benchmark.
- (b) Compute the Limit Comparison Test ratio.
- (c) Classify the series and give an independent direct lower comparison.

Question 2 – Solution

Step 1: Select a benchmark.

The leading-term quotient is $2n/n^2 = 2/n$, so choose $b_n = 1/n$. All terms are positive.

Step 2: Compute the ratio limit.

$$L = \lim_{n \rightarrow \infty} \frac{(2n+1)/(n^2+1)}{1/n} = \lim_{n \rightarrow \infty} \frac{2n^2+n}{n^2+1} = 2.$$

Because $0 < L < \infty$ and the harmonic series diverges, the given series diverges.

Step 3: Check the direction directly.

For $n \geq 1$, $n^2 + 1 \leq 2n^2$ and $2n + 1 \geq 2n$, so

$$\frac{2n+1}{n^2+1} \geq \frac{2n}{2n^2} = \frac{1}{n}.$$

A series larger than the divergent harmonic series must diverge.