

Question 10

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}.$$

- (a) Verify the domain and all Integral Test hypotheses for $f(x) = 1/[x\sqrt{\ln x}]$.
- (b) Evaluate the improper integral using $u = \ln x$.
- (c) Determine convergence or divergence.
- (d) Place this series within the family $\sum 1/[n(\ln n)^p]$ and state the threshold value of p .

Question 10 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/[x\sqrt{\ln x}]$ for $x \geq 2$. It is continuous and positive. Its denominator $x\sqrt{\ln x}$ is increasing, so f is decreasing. Equivalently,

$$f'(x) = -\frac{2\ln x + 1}{2x^2(\ln x)^{3/2}} < 0.$$

Step 2: Evaluate the improper integral.

Put $u = \ln x$, so $du = dx/x$:

$$\int_2^b \frac{dx}{x\sqrt{\ln x}} = \int_{\ln 2}^{\ln b} u^{-1/2} du = 2\sqrt{\ln b} - 2\sqrt{\ln 2}.$$

This tends to infinity, so the series diverges by the Integral Test.

Step 3: Identify the threshold.

More generally,

$$\sum \frac{1}{n(\ln n)^p}$$

converges for $p > 1$ and diverges for $p \leq 1$. Here $p = 1/2$, which lies on the divergent side.