

Question 8

Consider

$$\sum_{n=2}^{\infty} \frac{\ln n}{n^2}.$$

- (a) Verify the Integral Test hypotheses for $f(x) = \ln x/x^2$ on $[2, \infty)$.
- (b) Evaluate the improper integral using integration by parts.
- (c) Determine convergence and derive two-sided bounds for R_N .
- (d) Explain how the result illustrates polynomial growth dominating logarithmic growth.

Question 8 – Solution

Step 1: Verify the hypotheses on the tail.

Let $f(x) = \ln x/x^2$ for $x \geq 2$. It is continuous and positive. Also

$$f'(x) = \frac{1 - 2 \ln x}{x^3} < 0 \quad (x \geq 2),$$

so it decreases.

Step 2: Integrate by parts.

Take $u = \ln x$ and $dv = x^{-2}dx$. Then $du = dx/x$ and $v = -1/x$:

$$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + C.$$

Hence

$$\int_2^\infty \frac{\ln x}{x^2} dx = \frac{\ln 2 + 1}{2} < \infty,$$

and the series converges.

Step 3: Estimate the remainder.

Since the tail integral from A equals $(\ln A + 1)/A$,

$$\boxed{\frac{\ln(N+1) + 1}{N+1} \leq R_N \leq \frac{\ln N + 1}{N}}.$$

Quadratic denominator growth dominates the logarithmic numerator.