

Question 7

Consider

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}.$$

- (a) Verify positivity, continuity, and decreasing behavior for $f(x) = 1/(x^2 + 4)$.
- (b) Use the Integral Test to classify the series, showing the arctangent antiderivative and improper limit.
- (c) Write explicit upper and lower bounds for the remainder after N terms.

Your remainder bounds should contain N and inverse tangent functions.

Question 7 – Solution

Step 1: Verify the hypotheses.

Define $f(x) = 1/(x^2 + 4)$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = -\frac{2x}{(x^2 + 4)^2} < 0,$$

so it decreases.

Step 2: Test the improper integral.

$$\int_1^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left[\arctan\left(\frac{x}{2}\right) \right]_1^\infty = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{1}{2} \right) < \infty.$$

Therefore the series converges.

Step 3: Estimate the remainder.

For $A > 0$,

$$\int_A^\infty \frac{dx}{x^2 + 4} = \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{A}{2} \right).$$

Consequently,

$$\frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N+1}{2} \right) \leq R_N \leq \frac{1}{2} \left(\frac{\pi}{2} - \arctan \frac{N}{2} \right).$$