

Question 6

Consider the rapidly decaying series

$$\sum_{n=1}^{\infty} ne^{-n^2}.$$

- (a) Verify that $f(x) = xe^{-x^2}$ satisfies the Integral Test hypotheses on $[1, \infty)$.
- (b) Evaluate the improper integral with a substitution and classify the series.
- (c) Derive upper and lower bounds for the remainder R_N .
- (d) Explain what the bound says about the speed of convergence.

Question 6 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = xe^{-x^2}$ for $x \geq 1$. It is continuous and positive, and

$$f'(x) = e^{-x^2}(1 - 2x^2) < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the improper integral.

With $u = x^2$ and $du = 2x \, dx$,

$$\int_1^\infty xe^{-x^2} \, dx = \frac{1}{2} \int_1^\infty e^{-u} \, du = \frac{1}{2e}.$$

Thus the series converges by the Integral Test.

Step 3: Bound the remainder.

Since $\int_A^\infty xe^{-x^2} \, dx = \frac{1}{2}e^{-A^2}$,

$$\boxed{\frac{1}{2}e^{-(N+1)^2} \leq R_N \leq \frac{1}{2}e^{-N^2}}.$$

The Gaussian factor makes the tail decrease exceptionally rapidly.