

Question 5

Consider the nested-logarithm series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)}.$$

- (a) Explain why starting at $n = 3$ makes the terms positive and well defined.
- (b) Verify the Integral Test hypotheses for the corresponding function.
- (c) Evaluate the improper integral using two successive logarithmic substitutions.
- (d) State the verdict and identify the slow growth expression responsible for it.

Question 5 – Solution

Step 1: Check the domain and hypotheses.

Start at $n = 3$ so that $\ln n > 1$ and $\ln(\ln n) > 0$. Let

$$f(x) = \frac{1}{x \ln x \ln(\ln x)} \quad (x \geq 3).$$

The denominator is positive, continuous, and strictly increasing, so f is positive, continuous, and decreasing.

Step 2: Perform two substitutions.

First let $u = \ln x$, $du = dx/x$; then let $v = \ln u$, $dv = du/u$:

$$\int_3^b \frac{dx}{x \ln x \ln(\ln x)} = \int_{\ln 3}^{\ln b} \frac{du}{u \ln u} = [\ln(\ln u)]_{\ln 3}^{\ln b}.$$

Thus the antiderivative grows as $\ln(\ln(\ln b))$, which tends to infinity.

Step 3: Conclude.

By the Integral Test,

$$\boxed{\sum_{n=3}^{\infty} \frac{1}{n \ln n \ln(\ln n)} \text{ diverges.}}$$

This is another borderline series: three nested logarithms still grow without bound.