

Question 4

Consider

$$\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}.$$

- (a) For $f(x) = x/(x^2 + 1)^2$, verify positivity, continuity, and decreasing behavior on $[1, \infty)$.
- (b) Evaluate the associated improper integral using an appropriate substitution.
- (c) Determine convergence and obtain upper and lower bounds for R_N .

Clearly distinguish the value of the integral from the value of the series.

Question 4 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = x/(x^2 + 1)^2$ for $x \geq 1$. It is continuous and positive. Differentiation gives

$$f'(x) = \frac{1 - 3x^2}{(x^2 + 1)^3} < 0 \quad (x \geq 1),$$

so it decreases.

Step 2: Evaluate the integral.

With $u = x^2 + 1$ and $du = 2x \, dx$,

$$\int_1^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \int_2^\infty u^{-2} \, du = \frac{1}{4}.$$

The integral converges, so the series converges.

Step 3: Bound the tail.

Since

$$\int_A^\infty \frac{x}{(x^2 + 1)^2} \, dx = \frac{1}{2(A^2 + 1)},$$

the remainder satisfies

$$\boxed{\frac{1}{2((N+1)^2 + 1)} \leq R_N \leq \frac{1}{2(N^2 + 1)}}.$$