

Question 3

Consider

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}.$$

- (a) Verify that $f(x) = 1/[x(\ln x)^2]$ is positive, continuous, and decreasing for $x \geq 2$.
- (b) Apply the Integral Test, showing the substitution and evaluation of the improper integral.
- (c) Derive two-sided Integral Test bounds for the remainder R_N .
- (d) Briefly compare the result with $\sum 1/(n \ln n)$.

Question 3 – Solution**Step 1: Verify the hypotheses.**

Define $f(x) = 1/[x(\ln x)^2]$ for $x \geq 2$. It is continuous and positive, and

$$f'(x) = -\frac{\ln x + 2}{x^2(\ln x)^3} < 0,$$

so it decreases.

Step 2: Evaluate the integral.

Let $u = \ln x$, $du = dx/x$:

$$\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_{\ln 2}^\infty u^{-2} du = \frac{1}{\ln 2} < \infty.$$

Hence the series converges. The extra logarithm power changes $\int du/u$ into the convergent p -integral $\int du/u^2$.

Step 3: Estimate the remainder.

$$\boxed{\frac{1}{\ln(N+1)} \leq R_N \leq \frac{1}{\ln N}}.$$

These bounds follow by integrating from $N+1$ and N , respectively.