

Question 2

Consider the logarithmic series

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

- (a) Explain why the series begins at $n = 2$, and verify all three Integral Test hypotheses for $f(x) = 1/(x \ln x)$.
- (b) Evaluate the associated improper integral using $u = \ln x$.
- (c) State the convergence or divergence conclusion and describe the growth rate of the truncated integral.

The n th-term test alone is not a complete solution.

Question 2 – Solution**Step 1: Verify the hypotheses.**

Let $f(x) = 1/(x \ln x)$ for $x \geq 2$. It is continuous and positive because $x > 1$. Moreover,

$$f'(x) = -\frac{\ln x + 1}{x^2(\ln x)^2} < 0,$$

so f is decreasing, and $f(n)$ equals the summand.

Step 2: Evaluate the improper integral.

With $u = \ln x$ and $du = dx/x$,

$$\int_2^b \frac{dx}{x \ln x} = \int_{\ln 2}^{\ln b} \frac{du}{u} = \ln(\ln b) - \ln(\ln 2).$$

As $b \rightarrow \infty$, $\ln(\ln b) \rightarrow \infty$. Thus the improper integral diverges.

Step 3: Apply the test.

All hypotheses hold, so the Integral Test gives

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \text{ diverges.}$$

The growth scale $\ln \ln b$ explains why divergence is extremely slow.