

Question 10

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the two-step telescoping cancellation in a finite partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 10 – Solution

Step 1: Decompose the gap-two denominator.

Solving $1/[n(n+2)] = A/n + B/(n+2)$ gives $A = 1/2$ and $B = -1/2$. Thus

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Step 2: Telescope the finite sum.

$$\begin{aligned} s_N &= \frac{1}{2} \left[\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \right) - \left(\frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{N+2} \right) \right] \\ &= \frac{1}{2} \left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2} \right). \end{aligned}$$

The cancellation forms two interlaced chains because the indices differ by two.

Step 3: Take the limit.

Both ending reciprocals tend to zero, so

$$S = \frac{1}{2} \left(1 + \frac{1}{2} \right) = \boxed{\frac{3}{4}}.$$

Step 4: Find the exact remainder.

$$R_N = S - s_N = \frac{1}{2(N+1)} + \frac{1}{2(N+2)}.$$

As a check, subtracting consecutive partial-sum formulas reproduces $1/[N(N+2)]$.