

Question 9

Evaluate $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$.

- (a) Differentiate the geometric generating function twice.
- (b) Explain why $n^2 = n(n-1) + n$ requires restoring a first-derivative contribution.
- (c) Derive a closed generating function and evaluate it at $x = 1/2$.

Question 9 – Solution**Step 1: Differentiate twice.**

For $|x| < 1$, $(1 - x)^{-1} = \sum_{n=0}^{\infty} x^n$. Two derivatives give

$$\sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3}.$$

Multiplying by x^2 produces

$$\sum_{n=2}^{\infty} n(n-1)x^n = \frac{2x^2}{(1-x)^3}.$$

Step 2: Restore the missing linear part.

Because $n^2 = n(n-1) + n$, add the first-derivative identity $\sum nx^n = x/(1-x)^2$:

$$\sum_{n=1}^{\infty} n^2 x^n = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}.$$

Step 3: Evaluate at the requested point.

Since $x = 1/2$ lies inside the interval of convergence,

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{(1/2)(3/2)}{(1/2)^3} = \boxed{6}.$$

Omitting the $\sum nx^n$ term would compute $n(n-1)$ rather than n^2 .