

Question 8

Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$.

- (a) Differentiate the geometric power series inside its interval of convergence.
- (b) Align the powers and evaluate at $x = 1/2$.
- (c) Find the finite partial sum and exact remainder.

Question 8 – Solution

Step 1: Begin with a generating function.

For $|x| < 1$,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Power series may be differentiated term by term at interior points, giving

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Step 2: Match and evaluate the desired series.

Multiplying by x yields

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}.$$

Since $|1/2| < 1$, substitution is valid:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-1/2)^2} = \boxed{2}.$$

Step 3: Quantify the finite approximation.

Differentiating the finite geometric identity gives

$$s_N = \sum_{n=1}^N \frac{n}{2^n} = 2 - \frac{N+2}{2^N}, \quad R_N = \frac{N+2}{2^N}.$$

The remainder is positive and tends to zero, so the partial sums approach 2 from below.