

Question 6

Evaluate $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$.

- (a) Explain why this selects exactly the odd-indexed terms of the Basel series.
- (b) Compute the even-indexed contribution by scaling $\zeta(2)$.
- (c) Subtract to obtain the odd-square sum and verify that the two pieces recombine correctly.

Question 6 – Solution**Step 1: Partition the Basel series.**

Every positive integer is either even or odd, and the terms are positive, so

$$\sum_{m=1}^{\infty} \frac{1}{m^2} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}.$$

Step 2: Evaluate the even contribution.

Using Euler's value $\zeta(2) = \pi^2/6$,

$$\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$$

Step 3: Subtract to isolate the odd terms.

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{\pi^2}{24} = \boxed{\frac{\pi^2}{8}}.$$

Step 4: Verify the decomposition.

The even portion is one-fourth of the full sum, while the odd portion is three-fourths. Their sum is $(1/4 + 3/4)\zeta(2) = \zeta(2)$, as required.