

Question 4

Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$.

- (a) Find the partial-fraction decomposition.
- (b) Display the cancellation in the N th partial sum.
- (c) Find the sum and exact remainder after N terms.

Question 4 – Solution**Step 1: Decompose one summand.**

Write $1/[n(n+1)] = A/n + B/(n+1)$. Multiplication by $n(n+1)$ gives $1 = A(n+1) + Bn$, so $A = 1$ and $B = -1$. Hence

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

Step 2: Telescope a finite sum.

$$\begin{aligned} s_N &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\ &= 1 - \frac{1}{N+1}. \end{aligned}$$

All interior reciprocal terms cancel; only the two boundary terms remain.

Step 3: Pass to the limit and measure the error.

Since $1/(N+1) \rightarrow 0$, the series sums to $S = 1$. The exact remainder is

$$R_N = S - s_N = \frac{1}{N+1}.$$

As a check, $s_N - s_{N-1} = 1/[N(N+1)]$, the original N th term.